Multirate sampled-data systems: computing fast-rate models

Jiandong Wang\textsuperscript{a}, Tongwen Chen\textsuperscript{a,}\textsuperscript{*}, Biao Huang\textsuperscript{b}

\textsuperscript{a}Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB, Canada T6G 2V4
\textsuperscript{b}Department of Chemical and Materials Engineering, University of Alberta, Edmonton, AB, Canada T6G 2G6

Received 12 November 2002; received in revised form 3 February 2003; accepted 2 April 2003

Abstract

This paper studies identification of a general multirate sampled-data system. Using the lifting technique, we associate the multirate system with an equivalent linear time-invariant system, from which a fast-rate discrete-time system is extracted. Uniqueness of the fast-rate system, controllability and observability of the lifted system, and other related issues are discussed. The effectiveness is demonstrated through simulation and real-time implementation.

© 2003 Elsevier Ltd. All rights reserved.

Keywords: Multirate systems; System identification; State-space models; Controllability and observability

1. Introduction

The term multirate sampled-data (MRSD) systems describes a common phenomenon existing in the industry that different variables are sampled at different rates for some reasons \cite{2}, e.g. a high-purity distillation column \cite{13}, a bioreactor \cite{8} and CCR octane quality control \cite{15}. Fig. 1 depicts a single-input and single-output (SISO) MRSD system, where \( G_c \) is a continuous-time linear time-invariant (LTI) and causal system with or without a time delay; \( H_{mh} \) is a zero-order hold with an updating period \( mh \) and \( S_{nh} \) is a sampler with period \( nh \). Here \( m \) and \( n \) are different positive integers, and \( h \) is a positive real number called the base period. Discrete-time signals \( u \) and \( y \) are the system input and output respectively; a continuous-time signal \( v_c \) is the unmeasured disturbance. Essentially, it is a linear periodically time-varying (LPTV) system \cite{11}, to which many system identification algorithms cannot apply directly. Under such a framework, Lu and Fisher \cite{18,19} used an output error method and a least-squares method to estimate intersample outputs based on the fast sampled inputs and slow sampled outputs. Doi and Phillips \cite{6} developed multiple parametric models for the multirate systems using least-squares algorithms proposed by Lu and Fisher \cite{19}. Verhaegen and Yu \cite{21} extended a multi-variable output error state space (MOESP) class of algorithms to identify \( p \) subsystems of an LPTV process with period \( p \). Gudi et al. \cite{8} generated frequent estimates of the primary output based on the secondary outputs and the regular measurement of inputs by an adaptive inferential strategy. Li et al. \cite{14} identified a fast single-rate model with period \( mh \) from multirate input and output data, with an assumption that \( m < n \). This work motivates us: Could we do better?

Doing better implies two things: first, a fast-rate model with period \( h \) instead of \( mh \) will be identified; second, a general MRSD system is treated without the assumption \( m < n \). Note that our objective includes that of Li et al. \cite{14}, since a model with period \( mh \) is readily obtained from a model with period \( h \). The improvement is significant: technically, we need to use additional conditions such as observability of lifted models and coprimeness of the integers \( m \) and \( n \) (to be clarified later); in terms of applications, the availability of the fast-rate model with period \( h \) broadens the choices for multirate control design; the relaxation of assumptions makes identification of fast-rate models for more general MRSD systems possible, e.g. the input updating rates for MIMO MRSD systems no longer have to be uniform or be faster than the output sampling rates.

The question states precisely as follows: For a sampling period \( h \), the unknown continuous-time system \( G_c \) has a discrete-time counterpart realized by
the step-invariant-transformation, \( G_d = S_h G_s H_n \), represented by a state-space model:

\[
D + C(zI - A)^{-1}B = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]

(1)

Given the multirate input-output data in Fig. 1, how to identify the so-called fast-rate system \( G_d \)? To answer this question, we start in Section 2 with using the lifting technique to associate such an LPTV system with an LTI system, the so-called lifted system. The uniqueness of recovering the fast-rate system from the lifted system is shown in Section 3. Section 4 analyzes controllability and observability of the lifted system, which are essential to identifiability issues. Section 5 presents time-domain and frequency-domain approaches to compute a fast-rate model. Section 6 illustrates the effectiveness of the proposed methods through two examples. All previous sections have focused on SISO systems for the sake of an easier presentation. Finally, we give a discussion on the extension into MIMO systems in the last section.

2. Lifting signals and systems

Henceforth, we will focus our discussion on the SISO MRSD system depicted in Fig. 1. Let \( \psi \) be a discrete-time signal defined on \( \mathbb{Z}_+ \), set of non-negative integers, and \( n \) be some positive integer. The \( n \)-fold lifting operator \( L_n \) is defined as the mapping from \( \psi \) to \( \psi \):

\[
\{ \psi(0), \psi(1), \ldots \} \rightarrow \begin{bmatrix} \psi(0) \\ \psi(1) \\ \vdots \\ \psi(n - 1) \\ \psi(2n - 1) \end{bmatrix}.
\]

We lift \( u \) by \( L_n \) into \( u_k \) and lift \( y \) by \( L_m \) into \( y_k \). The disturbance \( v \) is fictitiously sampled into \( v \) with period \( nh \), same as the output sampling period, and \( y \) is lifted by \( L_m \) into \( y \) (see Fig. 2). Thus, \( u \), \( y \) and \( y \) share the same period \( mnh \), and form a discrete-time LTI system [7]:

\[
y = G_d u + v.
\]

(2)

Here \( G_d \) is the so-called lifted system from \( y \) to \( y \); it has a state-space representation by matrices \( A, B, C \) and \( D \), which are related to \( A, B, C \) and \( D \) of (1) as shown in Chen and Qiu [2]:

\[
\begin{bmatrix} A \mid B \\ C \mid D \end{bmatrix} := \begin{bmatrix} A^m & \sum_{i=0}^{m-1} A^i B & \sum_{i=0}^{m-1-m} A^i B & \cdots & \sum_{i=0}^{m-1} A^i B \\ C & D_{00} & D_{01} & \cdots & D_{0,n-1} \\ CA^n & D_{10} & D_{11} & \cdots & D_{1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{mn-n} & D_{m-1,0} & D_{m-1,1} & \cdots & D_{m-1,n-1} \end{bmatrix}
\]

(3)

where

\[
D_{ij} = D\chi_{[i,j]m}(\text{in}) + \sum_{r=j}^{i-1} CA^{in-r} B\chi(0, \text{in})(r)
\]

and a characteristic function on integers is defined:

\[
\chi_{[a,b]}(r) = \begin{cases} 1, & a \leq r < b \\ 0, & \text{otherwise} \end{cases}
\]

A noise model can be used to further describe the characteristics of the noise term \( v \) in (2), but it is not within our current objective. Hence, we adopt an output error model structure, since for open-loop systems, output error models will give consistent estimates, even if the additive noise is not white [16]. An innovation form of the state-space model with the Kalman filter gain \( K = 0 \) represents the overall lifted system:

\[
\begin{align*}
\dot{x} &= Ax + Bu + K \varepsilon, \\
y &= Cx + Du + \varepsilon.
\end{align*}
\]

Here \( \varepsilon \) is a white noise vector, and \( x \) is a state vector. If \( p \) is the order of \( G_d \), then the dimensions of \( A, B, C, D \) are \( p \times p, p \times 1, 1 \times p, \) and \( 1 \times 1 \), respectively, and those of \( A, B, C, D, K \) are \( p \times p, p \times n, m \times p, m \times n, \) and \( p \times m \), respectively. Note that \( A \) and \( A \) share the same dimension.

For illustration, consider Example 1 of Section 6 later, where \( m = 3, n = 2 \). Then (3) becomes

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^6 & A^5B + A^4B & A^3B & A^2B + AB + B \\ C & D & 0 \\ CA^2 & CAB + CB + D & 0 \\ CA^4 & CA^3B + CA^2B + CAB & CA^3B + CAB + CB + D \end{pmatrix}
\]

(4)
3. Uniqueness of fast-rate systems

Before starting the exploration of recovering the fast-rate system \( G_d \) from the lifted system \( G_{ld} \), a question arises naturally: Is the recovery unique? The answer is affirmative if and only if \( m \) and \( n \) are coprime. We observe from Fig. 2:

\[
G_{nd} = L_m S_{nh} G_c H_{mh} L_n^{-1} = L_m S_n (S_h G_c H_h) H_m L_n^{-1} \\
= L_m S_n G_d H_m L_n^{-1},
\]

by properties \( S_{nh} = S_n S_h \) and \( H_{mh} = H_h H_m \), where \( S_n \) is the discrete-time down-sampler by a factor of \( n \), and \( H_m \) is the discrete-time zero-order hold by a factor of \( m \). Since the lifting is one-to-one, the problem of recovery is the discrete-time zero-order hold by a factor of \( G_d \).

The time invariance of \( G_d \) and the definition of \( S_n \) imply

\[
\mu (im + jn) + \mu (im + jn + 1) + \cdots + \mu (im + jn + m - 1) = 0, \forall i, j.
\]

(6)

Since \( m \) and \( n \) are coprime, there exist integers \( m' \) and \( n' \) such that \( mm' + nn' = 1 \). Thus, for any \( k \), there always exist \( i = km' \) and \( j = kn' \) in (6) to get \( im + jn = k \). Hence,

\[
\mu (k) + \mu (k + 1) + \cdots + \mu (k - m - 1) = 0, \forall k.
\]

(7)

By causality of \( \mu (k) \), (7) implies that \( \mu (k) = 0, \forall k \), e.g. if \( k = -(m-1) \), then \( \mu (0) = 0 \); if \( k = -(m-2) \), then \( \mu (1) = 0 \) and so on. Hence, \( G_d = 0 \).

The necessity is proved as follows. If \( m \) and \( n \) are not coprime, there exists a common factor \( k: m = km', n = kn' \), where \( m' \) and \( n' \) are coprime. It follows from (5) that \( S_n G_d H_m = S_n G_{kd} H_{nh} \) where \( G_{kd} = S_h G_c H_{kh} \) i.e. a discrete-time counterpart of \( G_c \) with period \( kh \). Thus, the mapping \( G_d \mapsto S_n G_{kd} H_m \) is not one-to-one, since the mapping \( G_{ld} \mapsto G_{kd} = S_h G_c H_{kh} \) is known to be not injective.

In the sequel, we assume that \( m \) and \( n \) are coprime in order to get a unique fast-rate system. Note that any common factor of \( m \) and \( n \) can be absorbed into \( h \).

4. Controllability and observability of lifted systems

For a state space system to be identifiable, the lifted system \( G_d \) generally needs to be controllable and observable [16]. If the continuous-time system \( G_c \) is controllable and observable and the sampling period is non-pathological, then the discrete-time system \( G_d \) is also controllable and observable [9], which is still valid if a continuous time delay exists. Francis and Georgiou [7] have proved that if \( G_d \) is stabilizable and detectable, and satisfies an additional condition (*): For every eigenvalue \( \lambda \) of \( A \), none of the \( mn-1 \) points

\[
\lambda e^{\frac{2\pi i k}{mn}}, k = 1, 2, \ldots, mn - 1
\]

is an eigenvalue of \( A \), then \( \left( A^{mn}, A'B \right) \) is stabilizable and detectable, and \( \left( CA', A^{mn} \right) \) is detectable, for any positive integer \( i \). Based on these, we reach:

**Proposition 2.** Assume \( A \) satisfies the condition (*). If \( (CA, A) \) is observable, so is \( (C, A') \); if \( (A, B) \) is controllable and \( A \) has no eigenvalues on the unit circle, \((A, B)\) is also controllable.

**Proof:** The first part follows with some trivial modifications from Francis and Georgiou [7] in which \((CA', A)\) was shown detectable. We prove the second part by showing that \( \left( A + \sum_{i=0}^{m-1} A'B \right) \) is controllable, i.e. all eigenvalues of \( A \) are controllable. Now each eigenvalue of \( A \) has the form \( \lambda^m \), where \( \lambda \) is an eigenvalue of \( A \). Define functions:

\[
g(s) := \frac{s^{mn} - \lambda^{mn}}{s - \lambda}, f(s) := \sum_{i=0}^{m-1} \lambda^i
\]

By non-pathological sampling, \( g(A) \) is invertible [3]. If \( A \) has no eigenvalues on the unit circle, then \( \sum_{i=0}^{m-1} \lambda^i \neq 0 \). Thus \( f(A) \) is invertible. Therefore,

\[
\text{rank} \left( \left( A^{mn} - \lambda^{mn} I \right) \sum_{i=0}^{m-1} A'B \right) = \text{rank}(\left( A - \lambda I \right) g(A) f(A) B) = \text{rank} \left( f(A) \left[ A - \lambda I \right] \left[ f^{-1}(A) g(A) \quad 0 \right] \right) = \text{rank}(\left( A - \lambda I \right) B) = p.
\]
where the second equality holds because \( f(A) \) is a polynomial of \( A \) so that the commutativity holds. Thus, \((A, B)\) is controllable.

If there exists a continuous time delay \( \tau \) larger than \( h \), then \( A \) has at least two poles at \( z = 0 \) [1]. Thus, the condition \((*)\) is not satisfied. Observability has been shown to be lost and a remedy is proposed by Li et al. [14], which is summarized below: First, we can identify an \( m \times n \) time-delay matrix \( \Gamma \) from \( u, y \) using a correlation analysis [16]:

\[
\Gamma = \begin{bmatrix}
  l_{00} & l_{01} & \cdots & l_{0,n-1} \\
  l_{10} & l_{11} & \cdots & l_{1,n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  l_{m-1,0} & l_{m-1,1} & \cdots & l_{m-1,n-1}
\end{bmatrix}
\]

where \( l_{ij} \) is the estimated time delay (a nonnegative integer) from the \( j \)th input \( u_j \) to the \( i \)th output \( y_i \), \( i = 0, 1, \cdots, m-1 \), and \( j = 0, 1, \cdots, n-1 \). The relation between \( l_{ij} \) and \( \tau \) is given by (see (23) in the Appendix):

\[
(l_{ij} - 1)mnh < \tau + jmh - inh \leq l_{ij}mnh. \tag{8}
\]

Second, there exists a one-to-one correspondence between \( \Gamma \) and a positive integer \( k \) such that \( \tau \) is estimated as (see the appendix for a proof):

\[
kh < \hat{\tau} \leq kh + h. \tag{9}
\]

Third, since \( m \) and \( n \) are coprime, there exist integers \( k_1 \) and \( k_2 \) such that

\[
k = k_1m + k_2n. \tag{10}
\]

Finally, we shift the measured input data, \( u_i[s] = u[i-k_1] \), and shift the measured output data, \( y_i[s] = y[i-k_2] \), so that, the time delay between \( u \) and \( y \) is not larger than \( h \). Hence, controllability and observability will be preserved.

We illustrate the idea by estimating the time delay of Example 1 of Section 6, where \( m = 3 \), \( n = 2 \) and \( h = 1 \) sec. \( \Gamma \) is estimated as:

\[
\hat{\Gamma} = \begin{bmatrix}
  2 & 2 \\
  1 & 2 \\
  1 & 1
\end{bmatrix}
\]

From inequalities (8), we have

\[
6 < \hat{\tau} \leq 12, \ 3 < \hat{\tau} \leq 9, \\
2 < \hat{\tau} \leq 8, \ 5 < \hat{\tau} \leq 11, \\
4 < \hat{\tau} \leq 10, \ 1 < \hat{\tau} \leq 7.
\]

These inequalities give \( 6 < \hat{\tau} \leq 7 \), i.e. \( k = 6 \) in (9). Next, we shift the measured input or/and output data accordingly to (10): \( 6 = 3k_1 + 2k_2 \), say, \( k_1 = 0 \) and \( k_2 = 3 \), i.e. shift \( y \) by time advancing by three samples: \( y[i] = y[i+3] \). Effectiveness is confirmed by estimating the time delay between \( u \) and \( y \) in the same way.

5. Fast-rate model computation

In this section, two approaches (time-domain and frequency-domain) are proposed to compute the fast-rate system \( G_d \) after the lifted system \( \hat{G}_d \) has been estimated.

5.1. Time-domain approach

Once \( \hat{G}_d \) is estimated, how to extract matrices \( A, B \) and \( C \)? Note \( D = 0 \) if \( G_e \) is causal. The difficulty lies in that in general \( A \) cannot be determined by taking the \( mth \) roots of \( A \). Once \( A \), an estimate of \( A \), is known, \( B \) and \( C \) can be determined as

\[
\hat{C} = C_1, \quad \hat{B} = \left( \sum_{i=0}^{m-1} \hat{A}^i \right)^{-1} B_n, \tag{11}
\]

where \( B \) and \( C \) in (3) are partitioned as

\[
\hat{B} = [B_1 \ B_2 \ \cdots \ B_n], \tag{12}
\]

\[
\hat{C} = \begin{bmatrix}
  C_1^T & C_2^T & \cdots & C_m^T
\end{bmatrix}^T. \tag{13}
\]

Here the dimensions of \( B_1, B_2, \cdots, B_n \) are \( p \times 1 \), and those of \( C_1, C_2, \cdots, C_m \) are \( 1 \times p \). Note that the proof of Proposition 2 shows the existence of the inverse in (11). Another way is to consider most involved matrices to compute \( B \) and \( C \) using the least-square method. For instance,

\[
\hat{C} = \Phi^{-1} \Phi_e^T \left[ \Phi_e \Phi_e^T \right]^{-1}
\]

where

\[
\Phi = [C_1 \ C_2 \ \cdots \ C_m],
\]

\[
\Phi_e = \begin{bmatrix}
  \hat{A}^n \ \cdots \ \hat{A}^{m-n}
\end{bmatrix}.
\]

We propose two approaches to compute \( A \). The first approach, the controllability and observability approach, is based on assumptions that \( A_{mh}, B_{mh} \) is controllable and \( (A, A_{mh}) \) is observable, where

\[
A_{mh} := A^m, \quad A_{mh} := A^n,
\]

\[
B_{mh} := B_n = \sum_{i=0}^{m-1} A^i B.
\]

Similar to the proof of Proposition 2, both assumptions can be shown to be valid if the conditions in
Proposition 2 are true. This approach consists of three steps:

Step 1: Given \( A \) and \( B \) partitioned in (12), (3) implies
\[
A_{mnh}^m = A, \\
B_{mh} = B_m, A_{mh}B_{mh} = B_{n-1}, \cdots, A_{mnh}^{m-1}B = B_1.
\]

Thus, \( A_{mnh}B_{mh} \) is known for any \( k \geq 0 \). We form the controllability matrix \( \Omega_c \) of \( (A_{mnh}, B_{mh}) \) and the shifted controllability matrix \( \Omega_c \):
\[
\Omega_c = \left[ B_{mh} A_{mh}B_{mh} \cdots A_{mn}^{p-1}B_{mh} \right], \\
\Omega = \left[ A_{mh}B_{mh} A_{mh}^2B_{mh} \cdots A_{mn}^{p}B_{mh} \right].
\]

Since \( A_{mh}\Omega_c = \Omega \) and the controllability assumption implies \( \Omega_c \) is full row rank, \( A_{mh} \) is uniquely determined by
\[
\hat{A}_{mh} = \Omega_{c}^{-T}(\Omega_{c}^{-1})^{-1}.
\]

Step 2: Given \( A \) and \( C \) partitioned in (13), (3) implies
\[
A_{mn}^m = A, \\
C = C_1, CA_{nh} = C_2, \cdots, CA_{mn}^{m-1} = C_m.
\]

Thus, \( CA_{nh}^k \) is known for any \( k \geq 0 \). We form the observability matrix \( \Psi_o \) of \( (C,A_{nh}) \) and the shifted observability matrix \( \Psi_o \):
\[
\Psi_o = \left[ C \right], \Psi = \left[ CA_{nh} \right], \cdots, \left[ CA_{mh}^{p-1} \right], \cdots, \left[ CA_{nh}^{p} \right].
\]

Since \( \Psi_oA_{nh} = \Psi \) and the observability assumption implies \( \Psi_o \) is full column rank, \( A_{nh} \) is uniquely determined by
\[
\hat{A}_{nh} = \left( \Psi_o^T\Psi_o \right)^{-1}\Psi_o^T\Psi.
\]

Step 3: Now, \( A_{mh} = A^m \) and \( A_{nh} = A^n \) are estimated. Since \( m \) and \( n \) are coprime, there exist two integers \( m', n' \) such that
\[
m' - mn' = 1.
\]

Thus, we have:
\[
(A_{mh})^{m'} = (A_{nh})^{n'}.
\]

If \( A \) is invertible,
\[
\hat{A} = \left( A_{mh}^{-1} \right)^{m'} \left( A_{nh}^{-1} \right)^{n'}.
\]

Remark 1. In practice, time delays usually exist. Therefore, \( A_{mh} \) or \( A_{nh} \) is singular or nearly singular indicated by a large condition number. The inverse will be replaced by a pseudo-inverse. Most often, a good estimate of \( A \) is achieved by using the reduced singular value decomposition to realize the pseudo-inverse [12].

Remark 2. This approach relies on the structure in (3) heavily. To achieve such a structure, a state-space model with coupled parameters [17], the so-called “grey-box” model can be used.

The second approach, the matrix roots approach, is based on a condition that \( A \) is diagonalizable, i.e.
\[
P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_p)
\]
where \( \lambda_1, \lambda_2, \cdots, \lambda_p \) are eigenvalues of \( A \) and \( P \) is the corresponding eigenvector matrix. Since \( A = A^{mn} \), \( A \) and \( A \) share same eigenvectors. If \( \rho_i = \alpha_i + j\beta_i \) is a pole of \( G_s \), then
\[
\lambda_i = e^{\rho_i} \Rightarrow \lambda_i = e^{\rho_{mh}} = e^{\rho_{mh}^\prime} = e^{\rho_{nh}}
\]
With an assumption \( |mnh\beta_i| < \pi \) for \( i = 1, \cdots, p \),
\[
A = P\text{diag}(\lambda_1^{m_1}, \lambda_2^{m_2}, \cdots, \lambda_p^{m_p})P^{-1}
\]
where \( \lambda_i^{m} \) is the \( n \)th principal root of \( \lambda_i \); if this assumption is not true, \( A \) can be found by searching through all \( mnh \) roots of \( A \).

5.2. Frequency-domain approach

As a powerful tool, the polyphase decomposition can be used to compute the transfer function of \( G_d \) after \( G_{dL} \) has been estimated. For ease of discussion, let us illustrate the idea through Example 1 of Section 6, whose configuration is the same as that depicted in Fig. 1 with \( m = 3, n = 2 \) and \( h = 1 \) sec.

With identities [3]
\[
H_{3h} = H_{6\,S_{6}}H_{3h}, S_{2h} = S_{2h}H_{6\,S_{6}}, L_{6}^{-1}L_{6} = I, L_{6}L_{6}^{-1} = I,
\]
\( G_{d} \) in (5) can be written as
\[
G_{d} = L_{3}S_{2h}G_{c}H_{3h}L_{2}^{-1} \\
= L_{3}S_{2h}H_{6\,S_{6}}G_{c}H_{6\,S_{6}}H_{3h}L_{2}^{-1} \\
= \left( L_{3}S_{2h}H_{6\,L_{6}^{-1}} \right) \left( L_{6}G_{d}L_{6}^{-1} \right) \left( L_{6}S_{6}H_{3h}L_{2}^{-1} \right).
\]
The transfer matrix of \( S_L \) and \( H_L \), in (14) are constant matrices [3]:

\[
S_L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}, \quad H_L = \begin{bmatrix} I & 0 \\ I & 0 \\ I & 0 \\ 0 & I \\ 0 & I \\ 0 & I \end{bmatrix}.
\] (15)

The transfer matrix of \( G_d \), in (14), represented in the polyphase decomposition form, is a Toeplitz matrix given by Khargonekar et al. [10]:

\[
\hat{g}_d = \begin{bmatrix} g_0(z) & z^{-1}g_5(z) & z^{-1}g_4(z) & z^{-1}g_3(z) & z^{-1}g_2(z) & z^{-1}g_1(z) \\ g_1(z) & g_0(z) & z^{-1}g_5(z) & z^{-1}g_4(z) & z^{-1}g_3(z) & z^{-1}g_2(z) \\ g_2(z) & g_1(z) & g_0(z) & z^{-1}g_5(z) & z^{-1}g_4(z) & z^{-1}g_3(z) \\ g_3(z) & g_2(z) & g_1(z) & g_0(z) & z^{-1}g_5(z) & z^{-1}g_4(z) \\ g_4(z) & g_3(z) & g_2(z) & g_1(z) & g_0(z) & z^{-1}g_5(z) \\ g_5(z) & g_4(z) & g_3(z) & g_2(z) & g_1(z) & g_0(z) \end{bmatrix}.
\] (16)

Here \( g_0(z), g_1(z), \ldots, g_5(z) \) are defined in the polyphase decomposition of \( G_d \) introduced by Davis [5]:

\[
\hat{g}_d(z) = g(0) + z^{-1}g(1) + z^{-2}g(2) + \cdots
\]

\[
= \left[ g(0) + z^{-6}g(6) + \cdots \right] + z^{-1}[g(1) + z^{-6}g(7) + \cdots]
\]

\[+ \cdots + z^{-5}[g(5) + z^{-6}g(11) + \cdots] \]

\[= g_0(z^6) + z^{-1}g_1(z^6) + \cdots + z^{-5}g_5(z^6). \] (17)

From (14), (15) and (16), the transfer matrix of \( G_d \) in the polyphase decomposition form is obtained by retaining the first, the third and the fifth rows and adding every three columns starting from the upper-left corner of (16):

\[
\hat{g}_d = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \\ G_{31} & G_{32} \end{bmatrix}
\begin{bmatrix}
 g_0(z) + z^{-1}g_5(z) + z^{-1}g_4(z) + z^{-1}g_3(z) + z^{-1}g_2(z) + z^{-1}g_1(z) \\
 g_2(z) + g_1(z) + g_0(z) + z^{-1}g_5(z) + z^{-1}g_4(z) + z^{-1}g_3(z) \\
 g_4(z) + g_3(z) + g_2(z) + g_1(z) + g_0(z) + z^{-1}g_5(z) 
\end{bmatrix}.
\] (18)

Thus, we can establish six linear equations from (18), written in a matrix form:

\[
[ I \ 0 \ 0 \ 0 \ z^{-1} \ z^{-1} ] [ g_0(z) \ g_1(z) \ g_2(z) \ g_3(z) \ g_4(z) \ g_5(z) ] = [ G_{11} \ G_{12} \ G_{21} \ G_{22} \ G_{31} \ G_{32} ] [ I \ 0 \ 0 \ 0 \ z^{-1} \ z^{-1} ].
\] (19)

If \( G_d \) is estimated from the input-output data, i.e. \( G_{11}, G_{12}, \ldots, G_{31} \) are known, then \( g_0(z), g_1(z), \ldots, g_5(z) \) are uniquely determined, since the \( 6 \times 6 \) matrix in (19) is nonsingular. Therefore, the transfer matrix of \( G_d \) is determined uniquely by (17):

\[
\hat{g}_d(z) = \hat{g}_0(z^6) + z^{-1}\hat{g}_1(z^6) + \cdots + z^{-5}\hat{g}_5(z^6).
\] (20)

The procedure for this concrete example can be generalized in a straightforward manner. However, this approach is sensitive to the noise/error and usually gives a high order model. We can realize this point through (20), where \( z^6 \) is the operator. Thus, the approach requires more efforts, e.g. model reduction may be required. Even so, it is still an insightful way especially for theoretical analysis, e.g. feasibility and uniqueness of getting a fast-rate model are illustrated very clearly in this example.

6. Examples

Before giving examples, we are in a position to discuss some practical issues in implementation of the proposed methods.

Causality constraints: Lifting causes a causality constraint, i.e. \( D \) in (3) is (block) lower triangular. How to identify a model with such a constraint? A modified sub-space identification algorithm was proposed by Li et al. [14]. As an easier alternative, a structured state-space model with free parameters [17] can be used to deal with the constraint. For instance, \( D \) in (4), where \( m=3 \) and \( n=2 \), will be parameterized as

\[
\begin{bmatrix} 0 & 0 \\ \times & 0 \\ \times & \times \end{bmatrix}.
\]

where \( \times \) marks an adjustable parameter. Note \( D=0 \) if \( G_s \) is strictly causal.

Model validation: It is straightforward to validate the estimated fast-rate models in simulation or in experiments, since the true system or the fast-rate sampled input and output data are accessible. However, in practice, this information is unknown, which means that the fast-rate model validation cannot be implemented directly. An intuitive way is to put the esti-
mated fast-rate model into the same framework as the original system and compare the estimated output with the measured output through some criteria, such as the one based on the mean square errors.

Example 1. For a system depicted in Fig. 3, take the process and noise model to be

\[ G_c(s) = \frac{1}{20s^2 + 4s + 1} e^{-5t}, \quad N_c(s) = \frac{1}{10s + 1}, \]

and \( m = 3, \quad n = 2, \quad h = 1 \text{ sec}. \) \( e_c \) is a white noise in \([-0.333, 0.333]\). The identification procedure is: First, we generate a low frequency random binary signal (RBS) in \([-1, 1]\) as the input signal \( u \); second, we estimate the time delay as 6 sec and shift the measured output data \( y \) by time advancing by three samples into \( y_s \), as described in Section 4; third, we lift \( u \) by \( L_2 \) and lift \( y_s \) by \( L_3 \) to form the lifted signals with a time delay no larger than \( h \); next, based on the lifted signals, we choose a 2nd order lifted model \( \hat{G}_d \) and compute a fast-rate model \( \hat{G}_{\hat{d}} \) with period \( h \); finally, the estimated time delay is incorporated. Fig. 4 compares step responses of the actual system \( G_d \) and the estimated fast-rate models \( \hat{G}_{\hat{d}} \). The models are obtained through the proposed approaches: the frequency-domain approach and two time-domain approaches, namely, the controllability and observability approach and the matrix roots approach. All of them achieve satisfactory results.

Usually, the matrix roots approach gives the best results, since the other two require some approximations, e.g. those introduced by model reduction in the frequency-domain approach and by pseudo-inverse in the controllability and observability approach. However, it needs a condition that \( A \) is diagonalizable to be fulfilled, which limits its application.

Example 2. An experiment\(^1\) is implemented on a pilot-scale process in the computer process control laboratory at the University of Alberta. It is a SISO system with the manipulated input \( u \) as the cold water valve position and the measured output \( y \) as the tank water level, whose time constant is estimated around 400 sec. \( u \) and \( y \) are represented by currents (mA), which have linear relationships with the physical units. Around the operating point \( u = 11 \text{ mA} \) and \( y = 10.3 \text{ mA} \), a low frequency RBS input with a limiting magnitude of 0.4 mA, a

\(^1\) Data and Matlab programs are available online: http://www.ee.ualberta.ca/~jwang/paper.html.
constraint for safety, is designed. The input updating period is 80 sec and the output sampling period is 120 sec. Thus, \( m = 2 \), \( n = 3 \) and \( h = 40 \) sec, a dual configuration to Example 1. With “cheap” data acquisition, we simultaneously measure the input and output every 40 sec, say, \( u_f \) and \( y_f \), to be used later for model validation. Following a similar procedure as in Example 1, we estimate a 2nd order fast-rate model with period 40 sec, using the matrix roots approach. To validate the model, we take \( u_f \) as the model input and estimate the model output, which is compared with \( y_f \) in Fig. 5. The model captures the process dynamics and steady states very well.

7. Conclusions

Before ending the paper, we give a brief discussion on the extension of the presented results into MIMO systems. Is it always possible to get a fast-rate discrete-time MIMO model with the base period \( h \) for a general multirate MIMO system depicted in Fig. 6 that has \( p \) inputs and \( q \) outputs with different sampling rates? From Proposition 1, we know that a fast-rate model with the base period \( h \) is possible to be computed uniquely if and only if pairs of the input and output rates, \( (m_i, n_j) \), are coprime for all \( i \) and \( j \). The output signals will be lifted by \( L_{j/n_i} \) and input signals by \( L_{j/m_i} \), where \( J \) is the least-common-multiple of \( (m_1, m_2, \ldots, m_p, n_1, n_2, \ldots, n_q) \). Thus, the sampling period of the lifted system is \( Jh \). Most results for SISO systems can be extended into MIMO systems with minor modifications. However, the data shifting is not always feasible, although time delays of MIMO systems can be estimated in the same way as described for SISO systems. Therefore, the extension into MIMO MRSD system needs additional caution and efforts.

In this paper, we studied how to estimate a fast-rate model for a general multirate sampled-data system under some mild conditions. The idea is to associate the multirate sampled-data system with an equivalent lifted
system, from which the fast-rate model is extracted. Some topics are still open, e.g. how exactly the noise would affect the estimation? how to deal with time delays in MIMO systems? These are left to future investigation.

Appendix

**Proposition 3.** There exists a one-to-one correspondence between \( \Gamma \) and a positive integer \( k \) such that \( \tau \) is estimated as \( kh < \tau \leq kh+h \).

Before proving it, we introduce an inequality. During one sample period \( mnh \), say \((mnh, (l+1)mnh)\), \( y_j \) is sampled at the time instant \((mnh+jmh)\) and \( y_j \) at \((mnh+inh)\), where \( l \) is a positive integer and \( u_j \), \( y_j \) are defined in Section 4. The actual time delay \( \tau_j \) from \( u_j \) to \( y_j \), incorporating the effect of lifting, is
\[
\tau_j = \tau + jmh - inh.
\]

Meanwhile, \( \tau_{ij} \) is estimated from \( l_{ij} \) within one sample period, i.e.
\[
(l_{ij} - 1)mnh < \tau_{ij} \leq l_{ij}mnh.
\]

Substituting (21) into (22), we get the relation between \( l_{ij} \) and \( \tau \):
\[
(l_{ij} - 1)mnh < \tau + jmh - inh \leq l_{ij}mnh.
\]

**Proof:** It suffices to show that as \( \tau \) varies one \( h \) interval, one and only one element in \( \Gamma \) will change (an idea from Sheng et al. [20]). The proof consists of two parts: The first part shows the “only” term by contradiction; the second part shows there must be one element in \( \Gamma \) changed.

Suppose two different elements in \( \Gamma \), say \( l_{ij,1} \) and \( l_{ij,2} \), change as \( \tau \) varies one \( h \) interval, e.g. from \( kh-h < \tau_1 \leq kh \) to \( kh < \tau_2 \leq kh+h \). From (23), we have
\[
(l_{ij,1} - 1)mnh < \tau_1 + j_1mh - i_1nh \leq l_{ij,1}mnh,
\]
\[
l_{ij,1}mnh < \tau_2 + j_1mh - i_1nh \leq (l_{ij,1} + 1)mnh.
\]

Note that the changing step cannot be larger than 1, for \( \tau \) varies only \( h \) interval. Eliminating \( \tau_1 \) and \( \tau_2 \) in (24) and (25) gives
\[
l_{ij,1}mnh - kh - h < j_1mh - i_1nh < l_{ij,1}mnh - kh + h.
\]

Since \((j_1m-i_1n)\) is an integer,
\[
j_1m - i_1n = l_{ij,1}mn - k.
\]

Similarly to \( l_{ij,2} \),
\[
j_2m - i_2n = l_{ij,2}mn - k.
\]

Eqs. (26) and (27) give
\[
(j_2 - j_1) = (l_{ij,2} - l_{ij,1})n - (i_1 - i_2)n/m.
\]

Since \( m, n \) are coprime and \(|j_2 - j_1| < n-1\), we conclude \( l_{ij,1} = l_{ij,2} \) and \( j_2 = j_1 \), which cause a conflict.

Now the second part, let \( \tau \) varies as:
\[
kmnh < \tau_1 \leq kmnh + h,
\]
\[
(k + 1)mnh < \tau_{mn+1} \leq (k + 1)mnh + h.
\]

Define \( \Delta = jmh-inh \). From (23), we assume
\[
(l_{ij} - 1)mnh < \tau_1 + \Delta \leq l_{ij}mnh,
\]
\[
Eqs. (28) and (30) imply
\[
(l_{ij} - 1 - k)mnh \leq \Delta \leq (l_{ij} - k)mnh - h,
\]
for \( \Delta \) is an integer multiple of \( h \). Thus, (29) and (31) give
\[
l_{ij}mnh < \tau_{mn+1} + \Delta \leq (l_{ij} + 1)mnh.
\]

Eq. (32) means that every element in \( \Gamma \) increases by one after \( mnh \) intervals, which implies that there must be one and only one element in \( \Gamma \) changing as \( \tau \) varies one \( h \) interval, considering the proved first part. The proposition then follows.

**References**


