Design of controller for a class of pendulum-like system guaranteeing dichotomy

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Abstract

In this paper the problems of controller design for a class of nonlinear pendulum-like control system with multiple equilibria guaranteeing dichotomy and gradient-like property of the closed-loop systems are investigated. By applying KYP-lemma and some results of positive real control the method of controller design based on linear matrix inequality is proposed.

Keywords: Dichotomy; Pendulum-like systems; Multiple equilibria; Linear matrix inequality

1. Introduction

Many nonlinear control systems with multiple equilibria are often encountered in practice and the corresponding problems need to be dealt with. Types of equilibrium sets may be diverse and every type of equilibrium set corresponds to a certain type of nonlinear function. The pendulum-like system is a class of important nonlinear system whose nonlinear function is periodic. Many electric and electronic systems such as the systems of phase synchronization (phase-locked loops) are some class of the pendulum-like systems. There appear new types of stability problems for nonlinear control systems with multiple equilibria which are not the same as the systems with single stationary point. The global properties of solutions for nonlinear systems with multiple equilibria are required to investigate. One of the important property of solution for the pendulum-like systems is dichotomy which means every bounded solution of system is convergent. The other is gradient-like property of solutions which means every solution of system is convergent.

Notations. For $n \times m$ matrix with real elements is denoted by $A \in \mathbb{R}^{n \times m}$. $A^T$ denotes the transpose of $A$. $A > 0$ (respectively, $A \geq 0$) means matrix $A$ is positive definite (positive semi-definite).
2. Preliminaries and problem formulation

Consider nonlinear system of differential equations
\[ \dot{x} = f(t,x) \quad (f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n). \] (1)

We will use the following definitions throughout the paper (Leonov et al., 1996).

**Definition 1.** A solution \( x(t) \) of (1) is said to be convergent if \( x(t) \to c \) as \( t \to +\infty \), where \( c \) is an equilibrium point of (1). We say that the solution \( x(t) \) is quasi-convergent if \( \text{dist}(x(t), A) \to 0 \) as \( t \to +\infty \), where \( A \) is the set of equilibria points of (1) and \( \text{dist}(x, A) \) denotes the distance from a point \( x \) to the set \( A \), i.e., \( \text{dist}(x, A) = \inf_{z \in A} |x - z| \).

**Definition 2.** Eq. (1) is said to be dichotomous if its every bounded solution is convergent. It is called quasi-dichotomous if its every bounded solution is quasi-convergent. Eq. (1) is said to be gradient-like if its every solution is convergent.

Consider a class of autonomous pendulum-like system with two outputs
\[ \dot{x} = Ax + b\sigma(s), \quad \sigma = cx, \quad v = c_0x, \] (2)
where \( A \in \mathbb{R}^{n \times n} \) is a constant real matrix with \( \det A = 0 \), \( b \in \mathbb{R}^n \), \( c \in \mathbb{R} \) are real vectors, \( c \neq c_0 \). The measurable output \( v \) is different from \( \sigma \), input to nonlinearity \( \phi \). \( \phi(\sigma) \) is continuous for all \( x \in \mathbb{R} \) and satisfies
\[ \phi(\sigma + A) = \phi(\sigma), \quad \phi(\sigma) \neq 0, \]
where \( A \) is a period of \( \phi(\sigma) \). Let \( P_2(s) = c(A - sI)^{-1}b \) be the transfer function of the linear part of system (2) from \( -\phi(\sigma) \) to \( \sigma \) which is nondegenerate, that is, \( (A,b) \) is controllable and \( (A,c) \) is observable. The pendulum-like feedback system (2) can be described by Fig. 1.

Since some bounded solutions of the pendulum-like feedback system (2) may not be convergent, that is, not be dichotomous. It is necessary to consider the synthesis problem which we will investigate in this paper, that is, to design a linear controller \( K(s) \) by using the measurable output \( v \) such that the closed-loop feedback system described by Fig. 2 is dichotomous or gradient-like. Note that the transfer functions \( P_1(s) \) and \( P_2(s) \) can be represented as
\[ P_1(s) = c_0c(A - sI)^{-1}b \] and \[ P_2(s) = c(A - sI)^{-1}b \] in Fig. 1 and 2.

The following three lemmas are from Leonov et al. (1996).

**Lemma 1.** Suppose that the transfer function \( P_2(s) \) of the linear part of (2) from \( -\phi(\sigma) \) to \( \sigma \) is non-degenerate and has no pure imaginary poles besides the zero pole of multiplicity one. Suppose also that
\[ \text{Re}[j\omega P_2(j\omega)] \neq 0 \quad \text{for all} \ \omega \in \mathbb{R} \]
and
\[ \lim_{\sigma \to \infty} \omega^2 \text{Re}[j\omega P_2(j\omega)] \neq 0. \] (3)
Then system (2) is quasi-dichotomous.

**Lemma 2.** If all requirements of Lemma 1 are satisfied and the matrix \( A \) has \( n - 1 \) eigenvalues with negative real parts then system (2) is dichotomous.

**Lemma 3.** Suppose that the conditions of Lemma 2 are fulfilled and
\[ \int_0^\Lambda \phi(\sigma) \, d\sigma = 0, \] (4)
where \( \Lambda \) is a period of \( \phi(\sigma) \). Then system (2) is gradient-like.

For system (2), since \( \det A = 0 \), \( P_2(s) \) can be expressed as
\[ P_2(s) = (1/s)G(s) \] and condition (3) of Lemma 1 is satisfied if
\[ \text{Re} \ G(i\omega) \neq 0 \quad \text{for all} \ \omega \in \mathbb{R} \cup \mathbb{C}. \] (5)
which is equivalent to the condition that there exists a scalar \( \alpha \in \mathbb{R}, \alpha \neq 0 \) such that
\[ \alpha G(i\omega) + \alpha G^*(i\omega) > 0, \quad \omega \in \mathbb{R} \cup \mathbb{C}. \] (6)
This means the transfer function \( \alpha G(s) \) is extended strictly positive real (ESPR) (Sun, Khargonekar, & Shim, 1994) and thus we can obtain the following result from Lemmas 1 and 2.

**Lemma 4.** Suppose that \( G(s) \) is non-degenerate and zero is not the zero of \( G(s) \). If there exists a scalar \( \alpha \in \mathbb{R}, \alpha \neq 0 \) such that \( \alpha G(s) \) is ESPR, then system (2) is dichotomous.
3. Design of controller guaranteeing dichotomy for a class of pendulum-like feedback system

In this section, we will provide a method of designing linear controller $K(s)$ such that the closed-loop feedback system defined by Fig. 2 is dichotomous.

The transfer function of linear part for the closed-loop feedback system from $-\phi(\sigma)$ to $\sigma$ defined by Fig. 2 is

$$P_{cl}(s) = P_2(s)(1 + K(s))^{-1}$$

$$= c[I + (A - sI)^{-1}bK(s)c_*]^{-1}(A - sI)^{-1}b.$$  \hspace{1cm} (7)

Since $\det A = 0$, we can write the transfer function \((A - sI)^{-1}b\) as

$$(A - sI)^{-1}b = \frac{1}{s} G_0(s) = \frac{1}{s}(C_0(sI - A)^{-1}B_0 + D_0),$$  \hspace{1cm} (8)

where $A_0 \in \mathbb{R}^{n \times n}$, $B_0 \in \mathbb{R}^{n \times 1}$, $C_0 \in \mathbb{R}^{1 \times n}$ and $D_0 \in \mathbb{R}^{1 \times 1}$.

Throughout the paper we assume that

(A1) \((A_0, B_0)\) is controllable and \((A_0, C_0)\) is observable.

(A2) Zero is not the zero of $G_0(s)$.

In order to make the closed-loop feedback system defined by Fig. 2 to be still a pendulum-like feedback system, we assume that the designed controller $K(s)$ contains a differential part, that is,

$$K(s) = sK_1(s),$$  \hspace{1cm} (9)

where $K_1(s)$ has a state space realization

$$K_1(s) = \begin{bmatrix} \frac{A_k}{c_k} & b_k \end{bmatrix}.$$  \hspace{1cm} (10)

Thus, the designed controller $K(s)$ can be expressed as

$$K(s) = c_k A_k (sI - A_k)^{-1}b_k + c_k b_k.$$  \hspace{1cm} (11)

From (7)-(10) the transfer function $P_{cl}(s)$ of linear part for the closed-loop feedback system defined by Fig. 2 from $-\phi(\sigma)$ to $\sigma$ can be rewritten as

$$P_{cl}(s) = \frac{1}{s} c(I + G_0(s)K_1(s)c_*)^{-1}G_0(s).$$  \hspace{1cm} (12)

It is obvious that the closed-loop system described by Fig. 2 is still a pendulum-like feedback system if $P_{cl}(s)$ is nondegenerate.

In order to derive conditions for the existence of controller the condition that $P_{cl}(s)$ is nondegenerate is required.

From (A1) and (A2) we know that if $G_{cl}(s) = c(I + G_0(s)K_1(s)c_*)^{-1}G_0(s)$ is nondegenerate and zero is not the zero of $c(I + G_0(s)K_1(s)c_*)^{-1}$, then $P_{cl}(s) = (1/s)G_{cl}(s)$ is nondegenerate.

From Lemma 4 and above statements we can give the following result.

**Theorem 1.** Suppose that (A1) and (A2) hold. If there exist a scalar $\alpha \in \mathbb{R}, \alpha \neq 0$ and a transfer function $K_1(s)$ defined by (10) such that $G_{cl}(s) = c(1 + G_0(s)K_1(s)c_*)^{-1}G_0(s)$ is nondegenerate, zero is not the zero of $c(I + G_0(s)K_1(s)c_*)^{-1}$ and $zG_{cl}(s)$ is ESPR, then the closed-loop feedback system defined by Fig. 2 is dichotomous, where $K(s)$ is defined by (11).

From Theorem 1, the problem of designing controller $K(s)$ such that the system defined by Fig. 2 is dichotomous is transformed into the problem of designing a strictly proper transfer function $K_1(s)$ such that $zG_{cl}(s)$ is ESPR. In the following we will transform the latter problem into the ESPR control problem.

In fact, $zG_{cl}(s) = \alpha(1 + G_0(s)K_1(s)c_*)^{-1}G_0(s)$ has a state space realization

$$zG_{cl}(s) = \begin{bmatrix} A_0 & -B_0c_k \\ b_kc_0 & A_k - b_kc_0D_0c_k & B_0 \\ xc_0 & -xcD_kc_k & xcD_0 \end{bmatrix}.$$  \hspace{1cm} (13)

thus $zG_{cl}(s)$ can be expressed as

$$zG_{cl}(s) = \mathcal{F}(G_0, K_0).$$  \hspace{1cm} (14)

where $\mathcal{F}(G_0, K_0)$ is low linear fractional transformation (Zhou & Doyle, 1998) in which $G_0(s)$ and $K_0(s)$ have state space realizations, respectively,

$$G_0(s) = \begin{bmatrix} A_0 & B_0 & -B_0 \\ xc_0 & xcD_0 & -xcD_0 \\ c_0 & c_0D_0 & 0 \end{bmatrix},$$

$$K_0(s) = \begin{bmatrix} A_k - b_kc_0D_0c_k & b_k \\ 0 & c_k \end{bmatrix}.$$  \hspace{1cm} (15)

Therefore, the problem of designing a strictly proper transfer function $K_1(s)$ such that $zG_{cl}(s)$ is ESPR can be transformed into the ESPR control problem, that is, for a plant $G_0(s)$, find a strictly proper controller $K_0(s)$ which is defined by

$$K_0(s) = \begin{bmatrix} A_k & b_kc_0 & b_k \\ 0 & c_k \end{bmatrix}.$$  \hspace{1cm} (16)

such that $zG_{cl}(s) = \mathcal{F}(G_0, K_0)$ is internally stable and ESPR. In this case, the matrices of designed transfer function $K_1(s)$ can be expressed as

$$b_k = b_k, \quad c_k = c_k, \quad A_k = A_k + b_kc_0D_0c_k.$$  \hspace{1cm} (17)

Thus we have,

**Theorem 2.** Suppose that (A1) and (A2) hold. If there exist a scalar $\alpha \in \mathbb{R}, \alpha \neq 0$ and a strictly proper transfer function $K_0(s)$ defined by (16) such that $\mathcal{F}(G_0, K_0)$ is internally stable and ESPR, then there exists a transfer function $K_1(s)$ defined by (10) and (17) such that $zG_{cl}(s)$ is ESPR. Furthermore, if zero is not the zero of $c(I + G_0(s)K_1(s)c_*)^{-1}$ and $G_{cl}(s)$ is non-degenerate, i.e., $(A_k, b_k)$ is controllable and $(A_k, c_k)$ is observable, then the closed-loop feedback system described by Fig. 2
is dichotomous, where \( K(s) \) is defined by (11) and

\[
A_s = \begin{bmatrix} A_0 & -B_0c_k \\ b_kc_sC_0 & A_k - b_kc_sD_0c_k \end{bmatrix},
\]

\[
b_s = \begin{bmatrix} B_0 \\ b_kc_sD_0 \end{bmatrix}, \quad c_s = \{cC_0 - cD_0c_k\}.
\]  

(18)

In terms of the remark of Theorem 4.1 in Sun et al. (1994) and Theorem 2, the following LMI conditions for existence of controller \( K(s) \) such that the closed-loop feedback system described by Fig. 2 to be dichotomous can be obtained.

**Theorem 3.** Suppose that (A1) and (A2) hold. Then there exist a scalar \( x \in R, x \neq 0 \) and a strictly proper transfer function \( K_0(s) \) defined by (16) such that \( \mathcal{F}_A(G_0, K_0) \) is internally stable and ESPR if and only if there exist positive definite matrices \( W_1 > 0, W_2 > 0 \) and matrices \( W_2, W_4 \) such that

(i) \( xW_0D_0 > 0 \),

(ii) \[
\begin{bmatrix}
A_0W_1 + W_2A_0^T - B_0W_2 - W_2^Tb_k^T \\
-xW_2c_k^T - W_2^TD_0c_k - b_k^T
\end{bmatrix} < 0,
\]

(iii) \[
\begin{bmatrix}
W_4b_0c_k + W_4c_0c_k & c_0^TW_4^T \\
W_4b_0 + W_4c_0c_k & -c_0^TW_4^T
\end{bmatrix} < 0,
\]

(iv) \( p(Y_LX_F) < 1 \),

where \( X_F = W_1^{-1} \) and \( Y_L = W_2^{-1} \).

Moreover, if (i)–(iv) hold, then the matrices of state space realization for \( \mathcal{K}_0(s) \) can be expressed as

\[
\bar{c}_k = F = W_2W_1^{-1}, \quad \bar{b}_k = -(I - Y_LX_F)^{-1}L,
\]

\[
L = W_3^{-1}W_4,
\]

\[
\bar{A}_k = A_0 - B_0F + (I - Y_LX_F)^{-1}Lc_sC_0 + A_{FL},
\]

where

\[
A_{FL} = -\{2xcD_0\}^{-1}[B_0 + (I - Y_LX_F)^{-1}Lc_sD_0]
\]

\[
	imes[xcC_0 - xcD_0F - B_0^TX_F] - (I - Y_LX_F)^{-1}
\]

\[
\times Y_LF^T[-B_0^TX_F - \{2xcC_0 - xcD_0F - B_0^TX_F\}]
\]

\[
+(I - Y_LX_F)^{-1}Y_LR_F(X_F),
\]

in which

\[
R_F(X_F) = (A_0 - B_0F)^TX_F + X_F(A_0 - B_0F)
\]

\[
+(2xcD_0)^{-1}\{xcC_0 - xcD_0F - B_0^TX_F\}^T
\]

\[
\times (xcC_0 - xcD_0F - B_0^TX_F)
\]

and \( A_k, b_k, c_k \) can be formulated by (17).

Furthermore, if \((A_s, b_s)\) is controllable, \((A_s, c_s)\) is observable and zero is not the zero of \( c(I + G_0(s)K_1(s)c_s)^{-1} \), then the closed-loop feedback system described by Fig. 2 is dichotomous, where \( K(s) \) is defined by (11) and \( A_s, b_s, c_s \) are formulated by (18).

The dichotomy property of the systems can avoid oscillation of all the bounded solutions, which is often required for many practical systems. The method given in Theorem 3 is numerically feasible and thus provides a useful tool of controller design guaranteeing dichotomy for some practical systems.

**Remark 1.** Assume that \( \varphi(\sigma) \) satisfies (4) of Lemma 3 and the conditions of Theorem 3 hold. Then there exists a controller \( K(s) \) such that the closed-loop feedback system described by Fig. 2 is gradient-like.

**Remark 2.** Note that a basic character of the pendulum-like system (2) is that the matrix \( A \) has a zero eigenvalue. From (8), the perturbations of \( A, b \) with property of zero eigenvalue for \( A \) can be considered as perturbations of \( A_0, B_0, C_0 \) and \( D_0 \). Since the solution of LMIs in Theorem 3 has robustness for \( A_0, B_0, C_0, D_0 \) and \( c \). Therefore, the results of the controller design in Theorem 3 remain valid when \( A_0, B_0, C_0, D_0 \) and \( c \) are perturbed a bit. That is, when \( A, b, c \) are perturbed a bit, the given results remain valid provided the perturbed matrices of \( A \) always contain a zero eigenvalue.

**Remark 3.** The problem of designing controllers such that the closed-loop systems are dichotomous or gradient-like is quite interesting when the nonlinearity \( \varphi(\sigma) \) is not scalar. In this case, we need to use Theorem 2.9.1 (Leonov et al., 1996) instead of Lemmas 1–3, where the frequency-domain inequality is complicated compared with (3) in Lemma 1. Therefore the problem of controller design can not be simply transformed into the extended strictly positive real problem but it can be possibly dealt with by using other techniques related to LMI. Because of complexity and the limitation of space for this paper the corresponding results will be discussed in future papers.

### 4. Numerical example

Let us consider the following autonomous pendulum-like system with multiple equilibria:

\[
\dot{x} = Ax + b \sin(\sigma), \quad \sigma = cx, \quad v = c_sx,
\]

(19)

where

\[
A = \begin{bmatrix}
-3 & -2 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad b = \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix},
\]

\[
c = [1.4231, 0.8462, 2.5], \quad c_s = [-1, 3, 5].
\]

The solution of (19) is not convergent while choosing initial value \( x_0 = [10, 2, -1]^T \) (see Fig. 3) that illustrates this
system is not dichotomous. We also note that the frequency condition (3) of Lemma 1 is not satisfied for system (19).

By using Theorem 3 and Remark 1, we can design a controller $K(s)$ as

$$K(s) = \frac{0.28502280291046s^2 - 0.19783132632985s}{s^2 + 9.83681508144511s + 33.93913741615829}$$

which ensures the closed-loop feedback system described by Fig. 2 is gradient-like. The minimal state space representation of the closed-loop feedback system from $-\phi(\sigma)$ to $\sigma$ is

$$\dot{x}_{cl} = A_{cl}x + b_{cl}\sin(\sigma), \quad \sigma = c_{cl}x,$$

(20)

where

$$A_{cl} = \begin{bmatrix}
-7.991427431967 & -42.020571734517 & -114.435134824429 & -85.683094202003 & 0 \\
1.000000000000 & 0 & 0 & 0 & 0 \\
0 & 1.000000000000 & 0 & 0 & 0 \\
0 & 0 & 1.000000000000 & 0 & 0 \\
0 & 0 & 0 & 1.000000000000 & 0 
\end{bmatrix},$$

$$b_{cl} = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 
\end{bmatrix},$$

$$c_{cl} = \begin{bmatrix}
\end{bmatrix}.$$

It is obvious that (20) is also pendulum-like system with multiple equilibria. Note that if we choose initial value $x_0 = [10 \ 2 \ -1 \ 1 \ 6]^T$, then the corresponding solution of system (20) is convergent to equilibrium $x_e = [0 \ 0 \ 0 \ 7.141941127973204]^T$, where $x_e$ is an eigenvector of $A_{cl}$ corresponding to zero eigenvalue and $\sigma_e = c_{cl}x_e = -10907.60969299320$ is a root of $\sin(\sigma)$. The behavior of the corresponding solution for the closed-loop feedback system (20) is illustrated in Fig. 4.
5. Conclusions

In this paper some conditions for the existence of controller for a class of pendulum-like system guaranteeing dichotomy and gradient-like property of the closed-loop feedback system are given. The proposed method of controller design based on LMI is effective.

References


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