FREQUENCY DOMAIN METHOD FOR THE
DICHOTOMY OF MODIFIED CHUA’S EQUATIONS

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Received April 2, 2004; Revised August 11, 2004

On condition of dichotomy, it is pointed out that in Lorenz and a kind of Rössler-like system chaotic attractors or limit cycles will disappear if nonlinearity of the product of two variables is replaced by some single variable nonlinearity, for example, nonlinearity of Chua’s circuit. Furthermore, an extended Chua’s circuit with two nonlinear functions is presented. By computer simulation it is shown that oscillating phenomena in the extended Chua’s circuit are richer than the single Chua’s circuit. The corresponding extension for smooth Chua’s equations is also considered. The effects of input and output coupling are analyzed for the extended Chua’s circuit.

Keywords: Frequency domain method; dichotomy; Rössler-like system; Lorenz system; extended Chua’s circuit.

1. Introduction

The study of Chaotic attractors has attracted a lot of researchers since Lorenz presented a simple three-dimensional chaotic system, see [Celikovski & Chen, 2002; Chua, 1994; Lorenz, 1963; Madan, 1993; Shil’nikov, 1993] and references therein. During the last four decades two large classes of chaotic systems were studied extensively, one is with nonlinearity of the product of two variables such as Lorenz and Rössler systems [Chen & Ueta, 1999; Li et al., 2002; Rössler, 1979], another is with single variable nonlinear functions such as Chua’s circuits with piecewise linear (PWL) nonlinearity [Chua et al., 1986; Inai et al., 2002; Kapitaniak et al., 1994] and smooth Chua’s equations with cubic nonlinearity [Tsai, 1991; Chua et al., 1986; Liao & Chen, 1998; Tsuneda, 2005]. In addition, Rössler equation with nonlinearity of the product of two variables was generalized to a class of Rössler-like equations in which only single variable nonlinearity is involved [Thomas, 1999]. Based on the concept of feedback circuits, remarkable labyrinth chaos was shown in [Thomas, 1999].

So far, different methods for chaos analysis and control have been established [Chen & Dong, 1998; Madan, 1993]. The various literature references show that the study of chaos is a universal problem. In the study of chaos, it is generally hard to present the existence of chaotic attractors in theory [Stewart, 2000]. Based on the harmonic balance principle, a practical approach for predicting chaos dynamics in nonlinear systems was presented in [Genesio & Tesi, 1992]. This method is practically applicable, but not completely exact. Therefore, it is interesting to know the nonexistence of chaotic attractors. Recently, by using the frequency domain method developed in [Huang, 2003; Leonov et al., 1996] the frequency domain conditions of dichotomy were established to analyze the nonexistence of chaotic attractors or limit cycles in Chua’s circuit or coupled Chua’s circuit [Duan et al., 2004a; Leonov et al., 1996]. And the study of dichotomy
is an important aspect in view of the existence and nonexistence of complex behavior in dynamical systems [Chua, 1998].

This paper is devoted to studying the effects of input and output coupling in an extended Chua’s circuit with two nonlinear functions. Chaotic systems with more than one nonlinear function were studied and remarkable chaotic phenomena were shown in [Thomas, 1999; Yalcin et al., 2002; Duan et al., 2004a]. According to the location of the equilibrium points in state space, interesting scroll grid attractors were presented in [Yalcin et al., 2002]. Here, we also pay attention to the effect of nonlinear coupling by establishing the condition of dichotomy. The rest of this paper is organized as follows. In Sec. 2, the basic concept and results in frequency domain method are introduced briefly. In Sec. 3, linearity are compared in a Rössler-like system and product of two variables and single variable nonlinearity are introduced briefly. In Sec. 3, nonlinearity of the system and Rössler-like system chaotic attractors are presented in [Yalcin et al., 2002].

According to the location of the equilibrium points in state space, interesting scroll grid attractors were presented in [Yalcin et al., 2002]. Here, we also pay attention to the effect of nonlinear coupling by establishing the condition of dichotomy. The rest of this paper is organized as follows. In Sec. 2, the basic concept and results in frequency domain method are introduced briefly. In Sec. 3, nonlinearity of the system and Rössler-like system chaotic attractors are presented in [Yalcin et al., 2002].

Section 7 concludes the paper.

2. Frequency Domain Method for Systems with Single Variable Nonlinearity

In this section, we briefly review a basic definition and two results in frequency domain method. Consider the following system,

\[ \dot{x} = f(t, x), \tag{1} \]

where \( f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous and locally Lipschitz continuous in the second argument. Suppose that every solution \( x(t; t_0, x_0) \) of system (1) with \( t_0 \geq 0 \) and \( x(t_0) = x_0 \in \mathbb{R}^n \) can be continued to \( [t_0, +\infty) \).

Definition 1 [Leonov et al., 1996]. Equation (1) is said to be dichotomous if every bounded solution is convergent to a certain equilibrium of (1).

Obviously, if a nonlinear system is dichotomous, then the existence of chaotic attractors or limit cycles is impossible.

By using the Yakubovich–Kalman theorem [Leonov et al., 1996], the following system was studied in [Duan et al., 2004a]:

\[ \frac{dy}{dt} = Ay + B\varphi(x), \tag{2} \]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, y = (y_1, \ldots, y_m)^\top, x = (x_1, \ldots, x_m)^\top = (y_1, \ldots, y_m)^\top, \varphi(x) = (\varphi_1(x_1), \ldots, \varphi_m(x_m))^\top. \)

Notice that \( x \) is a part of \( y \), obviously Chua’s circuits with PWL nonlinearity, smooth Chua’s equations with cubic nonlinearity and Rössler-like systems studied in [Thomas, 1999] can be viewed as special cases of (2).

Viewing \( dx/dt \) as the output and \( \varphi(x) \) as the input, then system (2) can be viewed as a multi-input and multi-output (MIMO) system. Let \( C \) be the matrix composed of the first \( m \) rows of \( A \), and \( R \) be the matrix composed of the first \( m \) rows of \( B \).

Then the transfer function from \( \varphi(x) \) to \( x \) is

\[ K(s) = C(sI - A)^{-1}B + R. \]

A simple frequency domain condition of dichotomy for system (2) was established in [Duan et al., 2004a].

Lemma 1. Suppose that \( A \) has no pure imaginary eigenvalues, \( (A, B) \) is controllable and \( (A, C) \) is observable. If (3) has isolated equilibria and there exist diagonal matrices \( \epsilon = \text{diag}(\epsilon_1, \ldots, \epsilon_m) > 0 \) and \( \kappa = \text{diag}(\kappa_1, \ldots, \kappa_m) \) such that the following frequency domain inequality holds:

\[ \text{Re}(sK(iw) + K^\top(iw)sK(iw))) \leq 0, \quad \forall w \in \mathbb{R}, \tag{3} \]

then system (2) is dichotomous.

This lemma can be used to analyse the nonexistence of chaotic attractors or limit cycles. One just needs to test the frequency domain inequality (3). Generally, it is convenient to test this inequality by solving matrix inequality. So we often express the condition (3) in the form of linear matrix inequality (LMI) by KYP Lemma in [Anderson, 1967; Ranzer, 1966].

Lemma 2. If \( (A, B) \) is controllable, then (3) holds if, and only if, there exists \( P = P^\top \) such that
we can view \( \dot{y}_3 \) as the output of (4). Let \( C = [0 \ 0 \ -5.7] \), then \( \dot{y}_3 = C y + y_1 y_3 + 0.2 \). But now one can see that \((A, C)\) is not observable. In order to meet the observability condition, we can adjust system (4) a little bit as (we will see the meaning of observability in the forthcoming discussion)

\[
\frac{dy}{dt} = A_1 y + B_1(y_1 y_3 + 0.2),
\]

(5)

where

\[
A_1 = \begin{pmatrix}
0 & -1 & -1 \\
1 & 0.2 & 0 \\
0 & 0 & -5.7
\end{pmatrix}, \quad B_1 = \begin{pmatrix}
0 \\
1 \\
0.001
\end{pmatrix}, -5.7.
\]

\(B_1 = B, B\) and \(y\) are given as above. Let \( C_1 = [0.001 \ 0 \ -5.7]\). Computer simulations show that system (5) has a chaotic attractor similar to the Rössler attractor, see Fig. 1. Here we call system (5) as a Rössler-like system.

3.2. The Lorenz system

The following Lorenz system was studied extensively [Chen & Ueta, 1999; Lorenz, 1963; Liu et al., 2002]. Chen’s attractor and Liu’s attractor were found in Lorenz-type systems,

\[
\frac{dy}{dt} = A_2 y + B_2(y_1 y_3, y_2 y_3)^*,
\]

(6)

where

\[
A_2 = \begin{pmatrix}
-10 & 10 & 0 \\
28 & -1 & 0 \\
0 & 0 & -\frac{8}{3}
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
0 & 0 \\
-1 & 0 \\
0 & 1
\end{pmatrix},
\]

see Fig. 2 for the typical Lorenz attractor.

3.3. PWL nonlinearity of Chua’s circuit

In the Rössler-like system and Lorenz system above, one can see some typical chaotic phenomena. But what will happen if we change the nonlinear functions in (5) and (6) into nonlinearity of Chua’s circuit? Are there chaotic phenomena or limit cycles again? In what follows, we discuss such kind of problems. First according to (5), the following system is obtained by substituting the nonlinear function \(y_1 y_3 + 0.2\) into PWL function,

\[
\frac{dy}{dt} = A_1 y + B_1 f(y_3),
\]

(7)
where $A_1, B_1$ are given as in (5), $f(y_3) = G_2y_3 + 0.5(G_1 - G_2)(|y_1 + 1| - |y_3 - 1|)$, $G_1$ and $G_2$ are parameters to be chosen.

Let $C_1 = (0.001 \ 0 \ -5.7), D_1 = 1$, then the transfer function from $f(y_3)$ to $\dot{y}_3$ in system (7) is $K_1(s) = C_1(sI - A_1)^{-1}B_1 + D_1 = (s^3 - 0.2s^2 + s/s^3 + 5.5s^2 - 0.139s + 5.6998)$. Obviously $(A_1, B_1)$ and $(A_1, C_1)$ are controllable and observable, respectively. And by testing the matrix inequality in Lemma 2, we know that $K_1(s)$ satisfies the frequency domain condition in Lemma 1. Therefore, for any parameters $G_1$ and $G_2$ if (7) has isolated equilibria, then (7) is dichotomous. That is, there exist no chaotic attractors or limit cycles. For example, take $G_1 = -0.67, G_2 = 1.27$, see Fig. 3 for the solution of (7) at the same initial value as given in Fig. 1. From Fig. 3, one can see that the solution is dramatically unbounded.

Furthermore, by Lemma 1, one knows that when PWL function $f(y_3)$ in (7) is substituted by any piecewise continuous nonlinear function $g(y_3)$,
the corresponding system is dichotomous if it has isolated equilibria.

Similarly, if we substitute the nonlinear functions $y_1y_1, y_2y_3$ in the Lorenz system (6) by nonlinearity of Chua’s circuit, then we can get the following system

$$\frac{dy}{dt} = A_2y + B_2(f_1(y_2), f_2(y_3))^T,$$

where $A_2$ and $B_2$ are given as in (6). $f_1(y_2) = G_{12}y_2 + 0.5(G_{11} - G_{12})(|y_2 + 1| - |y_2 - 1|)$, $f_2(y_3) = G_{22}y_3 + 0.5(G_{21} - G_{22})(|y_3 + 1| - |y_3 - 1|)$.

Let $C_2$ be the matrix composed of the last two rows of $A_2$, $D_2$ be the matrix composed of the last two rows of $B_2$. Then the transfer function from $(f_1(y_2), f_2(y_3))^T$ to $(\dot{y}_2, \dot{y}_3)^T$ in system (8) is

$$K_2(s) = C_2(sI - A_2)^{-1}B_2 + D_2$$

$$= \begin{pmatrix} -s^2 - 10s & 0 \\ s^2 + 11s - 270 & s \\ 0 & s + \frac{N}{3} \end{pmatrix}.$$
Fig. 3. The solution of (7) with initial value \( y(0) = [-1, 0, 0]^T \). (a) \( y_1(t), y_2(t), y_3(t) \), and (b) \( y_1, y_2, y_3 \) space.

Obviously, \((A_2, B_2)\) and \((A_2, C_2)\) are controllable and observable, respectively. And by testing the matrix inequality in Lemma 2, we know that \( K_2(s) \) satisfies the frequency domain condition in Lemma 1. Therefore, for any parameters \( G_{11}, G_{12}, G_{21}, \) and \( G_{22} \) if (8) has isolated equilibria, then (8) is dichotomous.

Remark 2. From above, one can see that chaotic phenomena disappear in Rössler-like and Lorenz systems after substituting nonlinearity of the product of different variables by PWL nonlinearity. Of course, after this substitution, the system equilibria and the interactions of variables are changed largely. This also indicates the complexity and importance of the interactions of different variables in chaotic systems [Thomas, 1999; Chua, 1998]. In addition, we can see that the frequency domain method here cannot be used to deal with nonlinearity of the products of different variables. For simplicity, here we only discuss PWL nonlinearity of Chua’s circuit, some other nonlinear functions in [Huang et al., 1996; Thomas, 1999; Yalcin et al., 2002] can be considered similarly.
4. An Extended Chua’s Circuit with Two Nonlinear Functions

Chua’s circuits were studied extensively [Madan, 1993]. Generally, Chua’s circuits can be viewed as single-input and single-output feedback nonlinear systems. Coupled Chua’s circuits studied in [Duan et al., 2004a; Imai et al., 2002] can be viewed as multi-input and multi-output feedback nonlinear systems. Chua’s circuits and coupled Chua’s circuits can all be viewed as Lur’e systems [Suykens et al., 1997; Suykens et al., 1998]. The effects of input and output coupling were studied in [Duan et al., 2004a] for coupled Chua’s circuit. Some linearly

Fig. 4. The extended Chua’s circuit.

Fig. 5. The solution of (9) with initial value $v(0) = (1, -1, 0.4)^T (G_{21} = 0, G_{22} = 0)$. (a) $v_1(t), v_2(t), i_3(t)$, and (b) $v_1, v_2, i_3$ space.
interconnected systems were studied in [Duan et al., 2004b]. Generally, the dimension of the coupled system in [Duan et al., 2004a] is 6. In order to study the effects of input and output coupling in lower dimensional systems, we give an extended Chua’s circuit with two nonlinear functions, see Fig. 4. Chaotic systems with more than one nonlinear function were studied and remarkable chaotic phenomena were shown in [Thomas, 1999; Yalcin et al., 2002]. Here, we also pay attention to the effect of nonlinear coupling by establishing the condition of dichotomy.

The differential equation of the extended Chua’s circuit in Fig. 4 is

\[ \dot{v} = Av + B(f_1(v_1), f_2(v_2))^*, \]  

where

\[
A = \begin{pmatrix}
-1 & \frac{1}{C_1 R_1} & 0 \\
\frac{1}{C_2 R_1} & -1 & \frac{1}{C_2} \\
0 & -1 & -R_0/L_1 \\
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
-1/C_1 & 0 & 0 \\
0 & -1/C_2 & 0 \\
0 & 0 & 0 \\
\end{pmatrix},
\]

\[ v = \begin{pmatrix}
v_1 \\
v_2 \\
i_3 \\
\end{pmatrix}. \]

Fig. 6. The solution of (9) with initial value \( v(0) = (1, -1, 0.4)^*(G_{21} = 0.1, G_{22} = -0.03) \). (a) \( v_1(t), v_2(t), i_3(t) \), and (b) \( v_1, v_2, i_3 \) space.
$f_1$ and $f_2$ are two nonlinear functions defined as

\[ f_1(x) = G_{12}x + 0.5(G_{11} - G_{12})(|x + 1| - |x - 1|), \]
\[ f_2(x) = G_{22}x + 0.5(G_{21} - G_{22})(|x + 1| - |x - 1|). \]

Obviously, if $G_{21} = 0$, $G_{22} = 0$, the extended Chua’s circuit is the canonical Chua’s circuit studied in [Madan, 1993]. According to the traditional references on Chua’s circuit, there is a chaotic attractor in system (9) with parameters $C_1 = -1/1.301814, C_2 = 1, R_1 = 1, L_1 = -1/0.0136073, R_0 = 0.02969968/0.0136073, G_{11} = 0.1690817, G_{12} = -0.4767822, G_{21} = 0, G_{22} = 0$, see Fig. 5.

As one can imagine, the solutions of (9) will change largely with different parameters $G_{21}, G_{22}$ of the second nonlinearity, see Figs. 6–9. From Figs. 6–9, one can see more oscillating phenomena in system (9) by choosing the function $f_2$. So the oscillating phenomena of the extended Chua’s circuit are much richer than the single Chua’s circuit. One can increase the

Fig. 7. The solution of (9) with initial value $v(0) = (1, -1, 0.4)^*$$(G_{21} = 0.31, G_{22} = -0.25)$. (a) $v_1(t), v_2(t), i_3(t)$, and (b) $v_1, v_2, i_3$ space.
Fig. 8. The solution of (9) with initial value \( v(0) = (1, -1, 0.4)^T \) \( (G_{21} = 2.98, G_{22} = -0.35) \). (a) \( v_1(t), v_2(t), i_3(t) \), and (b) \( v_1, v_2, i_3 \) space.
complexity of behavior in simple systems systematically [Yalcin et al., 2002]. Of course, the extension of nonlinearity here can be used to consider other chaotic systems such as smooth Chua’s equations [Khibnik, 1993] and Rössler-like systems [Thomas, 1999].

5. Smooth Chua’s Equations

In contrast to PWL nonlinearity in Chua’s circuit, smooth Chua’s circuits with cubic nonlinearity were also studied thoroughly [Khibnik et al., 1993; Huang et al., 1996; Liao & Chen, 1998; Tsuneda, 2005]. The choice of a cubic nonlinearity has several advantages over a piecewise linear one. It does not require absolute-valued functions and it is smooth, which is desirable from a mathematical perspective. Moreover, almost all phenomena found in the PWL version also exist in cubic version. In what follows, we extend Chua’s circuit with cubic nonlinearity as above and see some other interesting oscillating phenomena.
The following smooth Chua’s circuit was studied in [Huang et al., 1996]
\[ \dot{v} = Av + Bx^3, \]  
where
\[ A = \begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}, \]
\[ B = \begin{pmatrix} -\alpha \\ 0 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \]

According to [Huang et al., 1996], there is a chaotic attractor in system (10) with parameters \( \alpha = 10, \beta = 16, c = -0.143 \), see Fig. 10.

We extend system (10) by adding another non-linearity as follows
\[ \dot{v} = A_1v + B_1[x^3, y^3]^T, \]

Fig. 10. The solution of (10) with initial value \( v(0) = (0.1, 0.1, 0.1)^T \). (a) \( x(t), y(t), z(t) \), and (b) \( x, y, z \) space.
where

\[ A_1 = \begin{pmatrix} -\alpha c & \alpha & 0 \\ 1 & -1 + f & 1 \\ 0 & -\beta & 0 \end{pmatrix}, \]

\[ B_1 = \begin{pmatrix} -\alpha & 0 \\ 0 & d \\ 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \]

\[ d \] and \[ f \] are new parameters to be chosen.

We can change the solutions of (10) largely by choosing \( d \) and \( f \), see Figs. 11–13.

Under the parameters \( \alpha = 10, \beta = 16, d = 0.5, f = 0.1 \) if we change \( c = -0.143 \) into \( c = -0.26 \), one can see the solution in Fig. 14.

From above, one can see that the solutions of smooth Chua’s equations can be changed largely by introducing another new nonlinearity. It is interesting to observe a systematic increase or decrease in the complexity of behavior in simple systems.

Fig. 11. The solution of (11) with initial value \( v(0) = (0.1, 0.1, 0.1)^T \)\((d = -0.2, f = 0.278)\). (a) \( x(t), y(t), z(t) \), and (b) \( x, y, z \) space.
Fig. 12. The solution of (11) with initial value $v(0) = (0.1, 0.1, 0.1)^T (d = 3.5, f = 0.1)$. (a) $x(t), y(t), z(t)$, and (b) $x, y, z$ space.
Fig. 13. The solution of (11) with initial value \( v(0) = (0.1, 0.1, 0.1)^T \) \( (d = 16, f = -0.1) \). (a) \( x(t), y(t), z(t) \), and (b) \( x, y, z \) space.
6. Input and Output Coupling

In this section, in order to increase or decrease the complexity of behavior in simple systems, we study the effects of a class of input and output coupling. Viewing \((\dot{v}_1, \dot{v}_2)\) as the output, \((f_1(v_1), f_2(v_2))\) as the input, system (9) can be viewed as an MIMO system. After some input and output coupling, a new system can be generated,

\[
\dot{v} = Av + B(f_1(\alpha_{11}v_1 + \alpha_{12}v_2),
\quad f_2(\alpha_{21}v_1 + \alpha_{22}v_2)),
\]

where \(A\) and \(B\) are matrices and \(\alpha_{ij}\) are constants. This system is an example of a coupled system, where the input and output are linked through nonlinear functions. The coupled system can exhibit more complex behavior than the original system, depending on the parameters and initial conditions.

Fig. 14. The solution of (11) with initial value \(v(0) = (0.1, 0.1, 0.1)^T, d = 0.5, f = 0.1, c = -0.26\). (a) \(x(t), y(t), z(t)\), and (b) \(x, y, z\) space.
where \( A, B, f_1, f_2 \) and \( v \) are given as in (9). Let \( M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \), \( z = M \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \). \( C, R \) be the matrices composed of the first two rows of \( A \) and \( B \), respectively. Here, we call \( M \) as a coupling matrix.

From (12), one can get
\[
\dot{z} = M \dot{v} + MR(f_1(a_{11}v_1 + a_{12}v_2), f_2(a_{21}v_1 + a_{22}v_2))^T.
\] (13)

The transfer function from \((f_1(z_1),f_2(z_2))^T\) to \( \dot{z} \) in systems (12) and (13) is \( \mathcal{M}K \in \text{systems (12) and (13).} \) By Lemma 1, one can get the following result for systems (12) and (13).

**Theorem 1.** Suppose that \( A \) has no pure imaginary eigenvalues, \( (A, B) \) is controllable and \( (A, C) \) is observable. If (12) has isolated equilibria and any matrix \( M \) such that the following frequency domain inequality holds:
\[
\text{Re}\{MK(iw) + K^*(iw)M^*MK(iw)\} \leq 0, \quad \forall w \in \mathbb{R},
\] (14)

then system (12) is dichotomous.

By using Schur complement, rewrite the matrix inequality in Theorem 3 as
\[
\begin{pmatrix}
PA + A^*P & PB \\
B^*P & 0
\end{pmatrix} + \begin{pmatrix}
C^* & 0 \\
0 & R^*M^*MR + \frac{1}{2} R^*M^* + \frac{1}{2} C^*M^* + C^*M^*MR
\end{pmatrix} \leq 0.
\] (16)

In terms of the solvability condition of nonstrict matrix inequality [Helmersson, 1995], if \( C \) is with full row rank, (16) holds if, and only if, there exists a scalar \( \beta > 0 \) such that
\[
\begin{pmatrix}
PA + A^*P & PB \\
B^*P & 0
\end{pmatrix} + \begin{pmatrix}
C^* & 0 \\
0 & R^*M^*MR + \frac{1}{2} R^*M^* + \frac{1}{2} C^*M^* + C^*M^*MR
\end{pmatrix} \leq 0.
\] (16)

If the coupling matrix \( M \) can be chosen deliberately, one can get a simplified result for system (12) by reducing the matrix variable \( \kappa \).

**Theorem 2.** Suppose that \( A \) has no pure imaginary eigenvalues, \( (A, B) \) is controllable and \( (A, C) \) is observable. If there exist diagonal matrices \( \epsilon = \text{diag}(\epsilon_1, \ldots, \epsilon_m) > 0 \) and any matrix \( M \) such that the following frequency domain inequality holds:
\[
\text{Re}\{MK(iw) + K^*(iw)M^*MK(iw)\} \leq 0, \quad \forall w \in \mathbb{R},
\] (15)

and system (12) has isolated equilibria, then system (12) is dichotomous.

Furthermore, by KYP Lemma in [Rantzer, 1996], one can also express the condition (15) in the form of linear matrix inequality (LMI).

**Theorem 3.** If \( (A, B) \) is controllable, then (15) holds if, and only if, there exists \( P = P^* \) such that
\[
\begin{pmatrix}
C^* & 0 \\
0 & R^*M^*MR + \frac{1}{2} R^*M^* + \frac{1}{2} C^*M^* + C^*M^*MR
\end{pmatrix} \leq 0.
\] (16)

In terms of the solvability condition of nonstrict matrix inequality [Helmersson, 1995], if \( C \) is with full row rank, (16) holds if, and only if, there exists a scalar \( \beta > 0 \) such that
\[
\begin{pmatrix}
PA + A^*P & PB \\
B^*P & 0
\end{pmatrix} \leq 0.
\] (16)

**Remark 3.** From the frequency domain inequalities (14) and (15), one knows that \( M = 0 \) is a trivial solution. When \( A \) has no pure imaginary eigenvalues, one can always take \( M = 0 \) such that system (12) is dichotomous. According to the method in [Helmersson, 1995], one can parameterize
all solutions $M$ to the matrix inequality (16) by solving the inequalities (17) and (18). So one can also consider the invariance of the property of dichotomy by the method here. By choosing the coupling matrix $M$, one can change the behavior of solutions of system (12) systematically.

From Fig. 9, one knows that system (9) is not dichotomous for $G_{21} = -0.26, G_{22} = -0.01$. But we can get $M=\begin{pmatrix} -12.522192 & 21.629156 \\ -2.880473 & 13.289164 \end{pmatrix}$ by solving the LMI in (16) such that the frequency domain inequality in Theorem 2 holds. So system (12) is
dichotomous when it has isolated equilibria for any piecewise continuous nonlinear functions (not only for nonlinearity of Chua's circuit). See Fig. 15 for the solution of (12) with the same initial value as given above.

From above we can see that oscillating solutions can be eliminated by choosing a suitable input and output coupling. On the other hand, we can also see some interesting oscillating phenomena under the input and output coupling

Fig. 16. (Continued)

Fig. 17. The solution of (12) with initial value $v(0) = (1, -1, 0, 4)^T, G_{21} = -2.98, G_{22} = -0.35$. (a) $v_1(t), v_2(t), i_3(t)$, and (b) $v_1, v_2, i_3$ space.
above. For example, take \( M = \begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix} \); we can see the oscillating solution in Fig. 16 for system (12).

Take \( M = \begin{pmatrix} 1 & -0.68 \\ 1.4 & 0.35 \end{pmatrix} \), we can see the oscillating solution in Fig. 17 for system (12).

Remark 4. In this section, only nonlinearity of Chua’s circuit is considered. Obviously, other nonlinear systems such as smooth Chua’s equations [Huang et al., 1996; Khibnik et al., 1993; Liao & Chen, 1998; Tsuneda, 2005] and Rössler-like systems [Thomas, 1999] can be studied similarly.

7. Conclusion

The frequency domain method established in [Duan et al., 2004a; Leonov et al., 1996] is used to discuss a class of feedback nonlinear systems. It is seen that a kind of Rössler-like system and the Lorenz system do not have chaotic attractors or limit cycles if the nonlinearity is replaced by some single variable nonlinearity. This also indicates that the frequency domain method here cannot be used to deal with nonlinearity of the products of different variables. Furthermore, an extended Chua’s circuit is presented. Although computer simulations show that much richer oscillating phenomena exist in the extended Chua’s circuit given here than in the single Chua’s circuit, the corresponding system after a suitable input and output coupling is dichotomous. The smooth Chua’s equations can be studied similarly. One can also see some new oscillating phenomena with the coupling here. It would be an interesting topic to increase or decrease the complexity of behavior in simple systems systematically.

Acknowledgments

The authors would like to thank the reviewers for their detailed and helpful comments which help improve the presentation of this paper. This work is supported by the National Science Foundation of China under grant 60204007, 60334030, 10472001.

References

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