Robust Dichotomy Analysis and Synthesis With Application to an Extended Chua’s Circuit

Ying Yang, Member, IEEE, Zhisheng Duan, and Lin Huang

Abstract—Dichotomy, or monostability, is one of the most important properties of nonlinear dynamic systems. For a dichotomous system, the solution of the system is either unbounded or convergent to a certain equilibrium, thus periodic or chaotic states cannot exist in the system. In this paper, a new methodology for the analysis of dichotomy of a class of nonlinear systems is proposed, and a linear matrix inequality (LMI)-based criterion is derived. The results are then extended to uncertain systems with real convex polytopic uncertainties in the linear part, and the LMI representation for robust dichotomy allows the use of parameter-dependent Lyapunov function. Based on the results, a dynamic output feedback controller guaranteeing robust dichotomy is designed, and the controller parameters are explicitly expressed by a set of feasible solutions of corresponding linear matrix inequalities. An extended OST of complex nonlinear dynamical systems can be analyzed and synthesized in a similar way.

Index Terms—Chaos control, dichotomy, linear matrix inequalities (LMIs), nonlinear systems, parameter-dependent Lyapunov function.

I. INTRODUCTION

MOST of complex nonlinear dynamical systems can be multistable, which means that when the system parameter values remain unchanged, the system state may be either chaotic or periodic. Such phenomenon often appears in electronic circuits, lasers, geophysical models, mechanical, and biological systems [1]–[3]. In many cases, multistability may change the performance and spoil the reliability of the system [4]. Therefore, a multistable system is often designed to be monostable or dichotomous. For a dichotomous system, the solution of the system is either unbounded or convergent to a certain equilibrium, thus periodic or chaotic states cannot exist in the system. Analysis of dichotomy is practically important and theoretically appealing. In the field of chaos study, it is generally hard to present the existence or nonexistence conditions of chaotic attractors in theory [5]. In [6], a practical approach for predicting chaos dynamics in nonlinear systems was presented based on the harmonic balance principle. This method is practically applicable, but not qualitatively exact. While [7]–[9] established frequency-domain condition of dichotomy with rigorous proof, which can be used as an alternative and effective way to detect whether a system has chaos or not and estimate regions for nonexistence of bounded oscillating solutions in the space of system parameters. From this point of view, the study of dichotomy is an important aspect in view of the existence and nonexistence of complex behavior in dynamical systems [10], [11].

Although a frequency-domain framework has been established in references [7]–[9], [12] within which the behavior of solutions with respect to entire set of equilibria was qualitatively analyzed, frequency-domain conditions derived within this framework often leads to heavy numerical procedures due to the requirement of frequency sweep. Moreover, it is hard to extend existing frequency-domain results to robust analysis and synthesis of systems with parametric or dynamic uncertainties. In this paper, the frequency-domain conditions of dichotomy for a class of nonlinear systems are revisited, and the LMI representations are derived. The results are further extended to the case where the linear part of the system is subject to real convex polytopic uncertainties, using the parameter dependent Lyapunov function method. This approach has been developed in [13]–[18] and proved to be efficient in dealing with polytopic uncertainties for linear uncertain systems either in continuous-time case [15], [16], [19] or in discrete-time case [18], [20]. While it seems so far that this method has not received much attention in nonlinear behavior analysis and synthesis, and there has been known few results on robust analysis and feedback design for nonlinear uncertain systems.

The contribution of this paper consists of the derivation of a new dichotomy condition characterized by linear matrix inequalities, and its extension to robust dichotomy analysis and synthesis for nonlinear systems with respect to real convex polytopic uncertainties. Dynamic output feedback controller existence conditions are established with the controller parameters explicitly expressed by a set of feasible solutions of corresponding linear matrix inequalities. Since the analysis and synthesis are carried out in time domain, the results obtained are more straightforward and numerically efficient than those in [7], [8], where the frequency-domain inequalities (FDIs) are used. Another novelty of the paper is that the system considered in this paper is allowed to have multiple equilibria, which is different from the conventional Lur’e systems with single equilibrium in classical absolute stability theory. Thus, using the method proposed in this paper, more global properties for a wider class of nonlinear systems, besides the absolute stability, such as Lagrange stability, well-posedness, complete stability, etc. can be analyzed and synthesized in a similar way. At the end of the paper, the validity and applicability of the
proposed method are demonstrated on an extended Chua’s circuit with symmetric nonlinear functions. This extended Chua’s circuits can have a richer nonlinear dynamic behavior than the canonical one due to the introduction of two additional parameters. When the system is subject to polytopic parameter uncertainties, the robust dichotomy can be achieved by designing appropriate controller. Comparison with the single Lyapunov function method illustrates the less conservatism of the proposed approach. It is also shown that dichotomy control is not only important in its own right, but also suggests a viable and effective way for chaos control in nonlinear circuits.

This paper is organized as follows: In Section II, some basic definitions and preliminary results are presented, while in Section III we present the main results of the paper. First, an LMI representation of dichotomy is presented and then extended to systems with polytopic uncertainties. Based on the results, a dynamic output feedback controller is designed to achieve robust dichotomy of the closed-loop system. The validity and applicability of the proposed approach is illustrated through a concrete nonlinear circuit and comparisons with the single Lyapunov function method in Section IV. Finally, concluding remarks are made in Section V.

Throughout the paper, the following notations are used: $\mathbb{R}^{n \times n}$ is the set of $n \times n$ real matrices. The superscript $T$ means transpose for real matrices, and $s$ means conjugate transpose for complex matrices. The matrix inequality $A > B(A \geq B)$ means that $A$ and $B$ are square Hermitian matrices and $A - B$ is positive (semi-) definite. For $n \times n$ matrix $A$, $\text{Re}(A) = A + A^*$, $\text{Re}(A) = (1/2)(A + A^*)$.

II. PRELIMINARIES

Consider the nonlinear system of differential equations

$$\dot{x} = f(t, x)$$

where $f : \mathbb{R}_{+} \times \mathbb{R}^{n} \to \mathbb{R}^{n}$ is continuous and locally Lipschitz continuous in the second argument. The following definitions are borrowed from [7].

**Definition 2.1:** The solution $x(t) = c$, where $c \in \mathbb{R}^{n}$ and $f(t, c) = 0$ for all $t \geq t_0$ is called a stationary solution or an equilibrium point.

**Definition 2.2:** A solution $x(t)$ of (1) is said to be convergent if $x(t) \to c$ as $t \to \infty$ where $c$ is an equilibrium point of (1).

**Definition 2.3:** Equation (1) is said to be dichotomic (or monostable) if every bounded solution is convergent.

**Remark 2.1:** From the definition of dichotomy, one can see that if a system is dichotomic, then its solution is either unbounded or convergent to an equilibrium point. For such systems, the existence of chaotic attractors or limit cycles is impossible.

In this paper, we consider the following nonlinear feedback system:

$$\begin{align*}
\dot{x} &= Ax + B\varphi(z) \\
\dot{z} &= Cx + D\varphi(z)
\end{align*}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{n \times m}$, $\varphi : \mathbb{R}^{m} \to \mathbb{R}^{m}$ is a vector valued function having the components $\varphi(z_i)$ with $z = (z_1, z_2, \ldots, z_m)^T$. We characterize system (2) by the transfer function of its linear part from the input $\varphi$ to the output $-z$

$$G(s) = C(A - sI)^{-1}B - D.$$  

The following assumptions are made on system (2).

**Assumption 2.1:** $(A, B)$ is controllable, $(A, C)$ is observable, and $A$ has no pure imaginary eigenvalues and the system has isolated equilibria.

**Assumption 2.2:** $\varphi_i : \mathbb{R} \to \mathbb{R}$ is piecewise continuously differentiable on $\mathbb{R}$ and there exist $\mu_{i1}$ and $\mu_{i2}$ such that

$$-\infty \leq \mu_{i1} \leq \frac{d\varphi_i}{dz_i} \leq \mu_{i2} < +\infty$$

for all $z_i \in \mathbb{R}$ where $\varphi_i(z_i)$ exists.

Let $\mu_{i1} = \text{diag}(\mu_{i11}, \ldots, \mu_{im1})$ and $\mu_{i2} = \text{diag}(\mu_{i12}, \ldots, \mu_{im2})$ with the numbers $\mu_{i1}$ and $\mu_{i2}$ from (3). Reference [7] gives the following frequency-domain conditions guaranteeing dichotomy of system (2).

**Lemma 2.1** [7]: Under the Assumptions 2.1 and 2.2, if there exist diagonal matrices $\epsilon = \text{diag}(\epsilon_1, \ldots, \epsilon_m) > 0$, $\tau = \text{diag}(\tau_1, \ldots, \tau_m) \geq 0$ and $\kappa = \text{diag}(\kappa_1, \ldots, \kappa_m)$ such that the following frequency-domain inequality holds:

$$\text{Re}\{\epsilon(j\omega) + \mu_{i1} G(j\omega) + j\omega \} \geq \kappa(j\omega) G(j\omega) \geq 0, \forall \omega \in \mathbb{R} \cup \{\infty\}.$$  

Then, system (2) is dichotomous.

**Remark 2.2:** Although system (2) has a form of Lur’e type, it allows to have multiple isolated equilibria, which is different from the typical Lur’e system with single equilibrium in absolute stability theory. Thus, the above result can be used to analyze nonlinear systems with multiple equilibria described by the differential (2), such as Chua’s circuit [21], phase synchronization systems [22], and some mechanical systems [23].

In addition to the known results mentioned above, we will need the following Kalman–Yakubovich–Popov (KYP) Lemma which establishes the equivalence between a frequency-domain inequality and a linear matrix inequality.

**Lemma 2.2** [24]: Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $M = M^T \in \mathbb{R}^{(n+m) \times (n+m)}$, with $\det((j\omega I - A) \neq 0$ for $\omega \in \mathbb{R}$ and $(A, B)$ controllable, the following two statements are equivalent:

1) $$(j\omega I - A)^{-1}B \ 	ext{and} \ (j\omega I - A)^{-1}B^T \text{ are positive definite}.$$

2) There exists a matrix $P = P^T \in \mathbb{R}^{n \times n}$ such that

$$M + \begin{bmatrix} A^TP + PA & PB \\ BP & 0 \end{bmatrix} \leq 0.$$

The corresponding equivalence for strict inequalities holds even if $(A, B)$ is not controllable.

III. MAIN RESULTS

A. LMI Conditions of Dichotomy

In this subsection, we introduce new alternative characterization of dichotomy for nonlinear systems with the form of (2).
The resulting conditions allow the use of parameter-dependent Lyapunov function when dealing with polytopic uncertainties.

**Theorem 3.1:** Under the Assumptions 2.1 and 2.2, system (2) is dichotomous if one of the following requirements is satisfied.

1) There exists a symmetric matrix $P = P^T$ and diagonal matrices $\epsilon > 0$, $\tau > 0$ and $\kappa$ such that
\[
\left[ \begin{array}{c|cc}
A^TP + PA + C^T\alpha(\epsilon, \tau)C & \gamma_1 + PB & Y_1 + P\beta
\
\hline
Y_1 & Y_2 & 2
\end{array} \right] \leq 0. \tag{5}
\]

2) There exists a symmetric matrix $P = P^T$, any nonsingular matrix $V$ and diagonal matrices $\epsilon > 0$, $\tau > 0$ and $\kappa$, such that
\[
\left[ \begin{array}{cccc}
-(V + VT) & VA + P & VB & V
\
A^TV^T + P & C^T\alpha(\epsilon, \tau)C - P & \gamma_1 & 0
\
B^TV^T & 0 & \gamma_1 & 0
\
VT & 0 & 0 & -P
\end{array} \right] \leq 0. \tag{6}
\]

3) There exist a symmetric matrix $P = P^T$, any nonsingular matrices $G$, $F$, and diagonal matrices $\epsilon > 0$, $\tau > 0$ and $\kappa$, such as shown by (7) at the bottom of the page, where
\[
\begin{align*}
\gamma_1 &= C^T\alpha(\epsilon, \tau)D + \frac{1}{2}C^T\kappa - \frac{1}{2}A^TC^T\beta(\tau) \\
\gamma_2 &= \Re \kappa D + D^T\alpha(\epsilon, \tau)D - \Re \beta(\tau)CB,
\end{align*}
\]
\[
\alpha(\epsilon, \tau) = \epsilon - \mu_1 \mu_2
\]
\[
\beta(\tau) = \tau(\mu_1 + \mu_2).
\]

**Proof:** The frequency inequality in Lemma 2.1 can be written as
\[
\Re G(j\omega) - G^*(j\omega)\alpha(\epsilon, \tau)G(j\omega) + \tau \omega^2 + \Re \beta(\tau)G_1(j\omega) \geq 0 \tag{8}
\]
with $G_1(s)$ defined as
\[
G_1(s) \triangleq sG(s) = CA(A - sI)^{-1}B - CB - sD
\]
and thus
\[
\Re G_1(j\omega) = \Re [CA(A - j\omega I)^{-1}B - CB]. \tag{9}
\]

From the definition of $G(s)$, we have
\[
G(j\omega) = C(A - j\omega I)^{-1}B - D. \tag{10}
\]

Substituting (9) and (10) into (8), it can be verified by some simple manipulations that (8) holds if
\[
\begin{align*}
&(j\omega I - A)^{-1}B^*C^T\alpha(\epsilon, \tau)C[(j\omega I - A)^{-1}B] \\
&\quad + \left[ D^T\alpha(\epsilon, \tau)C - \frac{1}{2}C^T\kappa - \frac{1}{2}\beta(\tau)CA \right] (j\omega I - A)^{-1}B
\end{align*}
\]

Let
\[
M = \left[ \begin{array}{cccc}
C^T\alpha(\epsilon, \tau)C & \gamma_1 & Y_1 & 2
\
\hline
\gamma_1 & \gamma_2 & 0 & 2
\end{array} \right]
\]
the above inequality can be rewritten into
\[
\left[ \begin{array}{cccc}
(j\omega I - A)^{-1}B^* & 0 & \gamma_1 & 0
\
\hline
0 & \gamma_1 & 0 & 0
\end{array} \right] M \left[ \begin{array}{cccc}
(j\omega I - A)^{-1}B^* & 0 & \gamma_1 & 0
\
\hline
0 & \gamma_1 & 0 & 0
\end{array} \right] \leq 0
\]
which is equivalent to
\[
M + \left[ \begin{array}{cccc}
A^TP + PA & PB & 0
\
\hline
B^TP & 0 & -P
\end{array} \right] \leq 0
\]
by applying Lemma 2.2. Namely, condition 1) of the theorem is satisfied. Note that the inequality in condition 2) can be written into the form of
\[
Q + \Re (\Sigma VW) \leq 0 \tag{11}
\]
with
\[
Q = \left[ \begin{array}{cccc}
0 & P & 0 & 0
\
\hline
P & C^T\alpha(\epsilon, \tau)C - P & \gamma_1 & 0
\
\gamma_1 & \gamma_2 & 0 & 0
\end{array} \right]
\]
\[
\Sigma = \left[ \begin{array}{cccc}
I & 0 & 0
\
\hline
0 & 0 & 0 & 0
\end{array} \right], \quad \Gamma = [-I \ A \ B \ I].
\]

The explicit bases of the nullspace of $\Sigma$ and $\Gamma$ are given as
\[
\Sigma^\perp = \left[ \begin{array}{cccc}
0 & I & 0 & 0
\
\hline
0 & 0 & I & 0
\end{array} \right], \quad \Gamma^\perp = \left[ \begin{array}{cccc}
A^T & I & 0 & 0
\
\hline
B^T & 0 & I & 0
\end{array} \right], \tag{12}
\]
Since the column space of $\Sigma$ and that of $\Gamma$ are linearly independent, applying the Projection Lemma for nonstrict inequality [25], [26] with respect to variable $V$ on (11), then (11) is solvable if and only if
\[
\Sigma^\perp Q\Sigma^\perp \leq 0, \quad \Gamma^\perp Q\Gamma^\perp \leq 0.
\]
Substituting (12) into the above inequalities, it can be easily verified that there exists $V$ such that (6) holds if and only if there exists $P$ such that (5) holds. According to the similar arguments, (7) can be written into the Hermitian form of (11) with respect
\[
\left[ \begin{array}{cccc}
-A^TG^T + P - F & GA + P - F^T & GB & 0
\
\hline
B^TG^T & B^T + \frac{1}{2}C^T\kappa - \frac{1}{2}\beta(\tau)CA & D^T\alpha(\epsilon, \tau)D - \Re \beta(\tau)CB & 0
\end{array} \right] \leq 0. \tag{7}
\]
to variable $G$ and there exists $G$ such that (7) holds if and only if there exists $P$ such that (5) holds.

**Remark 3.1.** In the above theorem, the frequency-domain conditions of dichotomy are transformed into the linear matrix inequalities conditions by using the KYP Lemma. The LMI representation makes it possible to extend the results to robust analysis and design of feedback controller guaranteeing dichotomy for systems with uncertainties.

**Remark 3.2.** It should be pointed out that condition 2) is based on the relaxation technique given in [13] and [15], and condition 3) is based on the technique in [14]. Although 1)–3) are equivalent when $(A, B, C, D)$ are given, 2) and 3) would render a less conservative conditions in case the matrices $A$ and $B$ are known to lie within the uncertainty polytope.

\[\Omega = \left\{ (A, B) \mid \begin{bmatrix} \sum_{i=1}^{m} \zeta_i (A^{(i)}, B^{(i)}) \end{bmatrix} \zeta_i \geq 0, \sum_{i=1}^{m} \zeta_i = 1 \right\} \quad (13)\]

where $(A^{(i)}, B^{(i)})$ is the $i$th vertex of the polytope $\Omega$, due to the freedom given by the slack variables and the fact that $P$ is allowed to be vertex-dependent. Moreover, 3) is generally less conservative than 2) due to an additional variable which makes 2) a special case of 3).

**Remark 3.3.** In [11], a design method of a static state feedback controller by using the condition 1) of Theorem 3.1 for a case of $z \in \mathbb{R}$ is proposed guaranteeing the dichotomy of system (2). Since the states of the system are not always measurable, in this paper, a dynamic output feedback controller is designed such that the system is dichotomous with real convex polytopic uncertainties.

**B. Robust Analysis**

In this subsection, we consider the robust dichotomy for system (2) when the matrices $A, B$ belong to the uncertainty polytope of (13).

**Definition 3.1.** System (2) is said to be robustly dichotomous if every bounded solution of the system is convergent for all $(A, B) \in \Omega$.

Since condition 3) of Theorem 3.1 is an LMI in $P, G, F, \epsilon, \tau$ and $\kappa$, it has to be satisfied only at each vertex of the polytope $\Omega$. Thus, we get the following result for robust dichotomy of system (2) corresponding to the uncertainty set (13).

**Theorem 3.2.** If there exist matrices $(P^{(i)}, G, F)$ with $P^{(i)} = P^{(i)T}$ and diagonal matrices $\epsilon > 0, \tau \geq 0$ and $\kappa$ such as shown by the equation at the bottom of the page, for $i = 1, \ldots, m$, where $\alpha(\epsilon, \tau), \beta(\tau)$ are as defined in Theorem 3.1, then system (2) under uncertainty set (13) is robustly dichotomous.

**Proof:** Denote the matrix on the left side of the inequality in the theorem by $\mathcal{L}(P)$, then it is deduced from the equation shown at the bottom of the page, that the uncertain system is robustly dichotomous.

**Remark 3.4.** By using PDLF method, the above result establishes an LMI-based criterion of robust dichotomy for system (2) with convex polytopic uncertainties. Since robust dichotomy can be proved through the use of a parameter-dependent Lyapunov matrix $\sum_{i=1}^{m} \zeta_i P_i$, this method is generally less conservative than the approaches based on single Lyapunov function for the whole uncertain domain $\Omega$.

The following corollary is a special case of Theorem 3.2 with $\tau = 0$ and thus $\alpha(\epsilon, \tau) = \epsilon, \beta(\tau) = 0$, which enables one using the result of Theorem 3.2 to deal with uncertainties in all system matrices characterized by

\[\Omega_1 = \left\{ (A, B, C, D) \mid \begin{bmatrix} \sum_{i=1}^{m} \zeta_i (A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}) \zeta_i \geq 0, \sum_{i=1}^{m} \zeta_i = 1 \right\} \quad (14)\]

and design feedback controller to ensure robust dichotomy for system. Here, $(A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)})$ is the $i$th vertex of the polytope $\Omega_1$.

**Corollary 3.1:** System (2) is robustly dichotomous under uncertainty set (14) if there exist symmetric matrices $P^{(i)} = P^{(i)T}$, any nonsingular matrices $G, F$ and diagonal matrices $\epsilon > 0, \kappa$ such that

\[
\begin{bmatrix}
-(G + G^T) & \Xi & 0 & 0 \\
\Xi^T & FA^{(i)} + A^{(i)T} F^T & FB^{(i)} + C^{(i)T} \kappa & 0 \\
B^{(i)T} G & B^{(i)T} F + \frac{1}{2} \kappa C - \frac{1}{2} \beta(\tau) C A^{(i)} & 0 & C^{(i)T} D \kappa \\
0 & 0 & 0 & -\epsilon^{-1}
\end{bmatrix} < 0
\]

where $\Xi = GA^{(i)} + P^{(i)} - F^T$.

**Remark 3.5.** Let $\delta \equiv \epsilon^{-1}$ be a new matrix variable in the above LMI. Since $\delta$ is not involved in any product term with system matrices, it can also be parameter-dependent, which helps reduce the degree of conservatism of the result.

\[
\begin{bmatrix}
-(G + G^T) & 0 & 0 & 0 \\
A^{(i)T} G^T + P - F & 0 & 0 & 0 \\
B^{i} F + \frac{1}{2} \kappa C - \frac{1}{2} \beta(\tau) C A^{(i)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \leq 0
\]

\[
\begin{bmatrix}
-(G + G^T) & 0 & 0 & 0 \\
A^{(i)T} G^T + P - F & 0 & 0 & 0 \\
B^{i} F + \frac{1}{2} \kappa C - \frac{1}{2} \beta(\tau) C A^{(i)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \leq 0
\]

\[
\begin{bmatrix}
-(G + G^T) & 0 & 0 & 0 \\
A^{(i)T} G^T + P - F & 0 & 0 & 0 \\
B^{i} F + \frac{1}{2} \kappa C - \frac{1}{2} \beta(\tau) C A^{(i)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \leq 0
\]
C. Controller Synthesis

In this subsection, we propose an LMI-based approach on designing a dynamic output feedback controller guaranteeing the robust dichotomy of system (2) under polytopic uncertainty set (14). Consider the nonlinear system

\[
\begin{align*}
\dot{x} &= Ax + B\varphi(y) \\
\dot{z} &= Cx + D\varphi(y)
\end{align*}
\] (15)

where \(\varphi\) satisfies (3), and \(A, B, C, D\) are under uncertainty set (14). We suppose the controller \(K(s)\) has the following state space realization:

\[
\begin{align*}
\dot{x}_k &= A_k x_k + B_k \dot{z} \\
\dot{y}_k &= C_k x_k + D_k \dot{z}
\end{align*}
\]

Then, the closed-loop system has the form of

\[
\begin{align*}
\dot{x}_{cl} &= A_{cl} x_{cl} + B_{cl} \varphi(y) \\
\dot{y}_{cl} &= C_{cl} x_{cl} + D_{cl} \varphi(y)
\end{align*}
\]

where

\[
\begin{align*}
x_{cl} &= \begin{bmatrix} x \\ x_k \end{bmatrix}, \quad A_{cl} = \begin{bmatrix} A & 0 \\ B_k C & A_k \end{bmatrix}, \quad B_{cl} = \begin{bmatrix} B \\ B_k D \end{bmatrix} \\
C_{cl} &= \begin{bmatrix} D_k C & C_k \end{bmatrix}, \quad D_{cl} = D_k D.
\end{align*}
\]

The closed-loop nonlinear feedback system is shown in Fig. 1 where \(G_{cl}(s) = C_{cl}(A_{cl}-sI)^{-1}B_{cl}-D_{cl} = K(s)G(s)\). Then, we are ready to give the following result which establishes the controller existence conditions for robust dichotomy of system (15) under polytopic uncertainty (14).

**Theorem 3.3:** There exists a dynamic feedback controller such that system (15) under uncertainty (14) is robustly dichotomous if there is a solution

\[
(G_{11}, G_{21}, G_{2}, F_{11}, F_{21}, A_{11}, A_{21}, S_A, S_B, S_C, S_D, \kappa, \delta, P_{11}^{(i)}, P_{12}^{(i)}, P_{22}^{(i)}, i \in \mathbb{N}, \lambda_1, \lambda_2)
\]

to

\[
\begin{bmatrix}
-(G_{11} + G_{11}^T) & -(G_{21} + G_{21}^T) & \Phi_{11} & \Phi_{12} & \Phi_{13} & 0 \\
-(G_{21}^T + G_{21}) & -(G_{21} + G_{11}^T) & \Phi_{21} & \Phi_{22} & \Phi_{23} & 0 \\
\Phi_{11}^T & \Phi_{21}^T & \Phi_{11} & \Phi_{12} & \Phi_{13} & S_{T_C} \\
\Phi_{12}^T & \Phi_{22}^T & \Phi_{21} & \Phi_{22} & \Phi_{23} & S_{T_C} \\
\Phi_{13}^T & \Phi_{23}^T & \Phi_{13} & \Phi_{23} & \Phi_{23} & \Phi_{34} & S_C \\
0 & 0 & \Phi_{34}^T & S_C & \Phi_{34}^T & \Phi_{34}^T & \Phi_{34}^T & \Phi_{34}^T
\end{bmatrix} < 0
\]

(16)

where

\[
\begin{align*}
\Phi_{11} &= G_{11} A^{(i)} + S_B C^{(i)} + P_{11}^{(i)} F_{11}^T \\
\Phi_{12} &= G_{A_{11}} A^{(i)} + F_{12} - F_{21}^T \\
\Phi_{13} &= G_{11} B^{(i)} + S_B D^{(i)} \\
\Phi_{21} &= G_{21} A^{(i)} + S_B C^{(i)} + P_{12}^{(i)} - \lambda_1 G_{21} \\
\Phi_{22} &= F_{12} + \lambda_2 G_{22} \\
\Phi_{23} &= G_{22} B^{(i)} + S_B D^{(i)} \\
\Phi_{31} &= F_{11} A^{(i)} + A^{(i)T} F_{11} + \lambda_1 S_B C^{(i)} + \lambda_1 C^{(i)T} S^T_B \\
\Phi_{32} &= \lambda_2 S_A + A^{(i)T} F_{12} + \lambda_2 C^{(i)T} S^T_B \\
\Phi_{33} &= F_{11} B^{(i)} + \lambda_2 S_B D^{(i)} + \lambda_2 C^{(i)} S D^T_D \\
\Phi_{34} &= \kappa D_{cl} F_{11} + \lambda_2 C^{(i)T} S_D^T D^T_D \\
\Phi_{41} &= \lambda_2 S_{A} + \lambda_2 S_{A} \\
\Phi_{42} &= F_{21} B^{(i)} + \lambda_2 S_{B} D^{(i)} + \lambda_2 C^{(i)} S_D^T D^T_D \\
\Phi_{51} &= \kappa D_{cl} F_{12} + \lambda_2 C^{(i)T} S_D^T D^T_D \\
\Phi_{52} &= \lambda_2 C^{(i)} S_D^T D^T_D \\
\Phi_{i1} &= \kappa D_{cl} F_{12} + \lambda_2 C^{(i)T} S_D^T D^T_D
\end{align*}
\]

and the controller is given by \(A_k = G_{21}^{-1} S_A, B_k = G_{21}^{-1} S_B, C_k = S_C, D_k = S_D\).

**Proof:** From Corollary 3.1, the closed loop system is robustly dichotomous if there exist \(P = P^T, \mu\), nonsingular matrices \(G, F\), and diagonal matrices \(\epsilon > 0\) and \(\kappa\) such that

\[
\begin{bmatrix}
-(G + G^T) & GA_{4} + P - F^T & GB_{4} \\
AT^G + P - F & FA_{4} + AT^G F & FB_{4} + C_{T}^{T} K \\
B_{cl} G^T & B_{cl} F + \kappa C_{cl} D_{cl} + D_{cl}^T D_{cl}^T & \delta
\end{bmatrix} < 0
\]

Write \(G\) and \(F\) as the following block matrices:

\[
G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \quad F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}
\]

where the dimensions of \(G_{11}, G_{11}\) and \(G_{22}, F_{22}\) are consistent with the ones of \(A\) and \(A_k\), respectively. For full-order controller design, we can assume that \(G_{11}\) and \(G_{22}\) are nonsingular without loss of generality. Multiplying the left and the right of (17) by \(\Pi = \text{diag}(X X T I T)\) and \(\Pi^T\), respectively, where \(X = \text{diag}(I G_{12} G_{22})\) yields (18), as shown at the bottom of the page, where \(\Theta_1 = -(XG_{12}^T + XG_{12}^T X^T), \Theta_2 = XGB_{4} + XFB_{4} + \kappa C_{cl} D_{cl} + D_{cl}^T D_{cl}^T \delta\), and \(\Theta_3 = \kappa D_{cl} D_{cl} + D_{cl}^T D_{cl}^T \theta\).
\[ \Theta_2 = XGAX^T + XPX^T - XFT^T X^T, \Theta_3 = XFA_\delta X^T + XA_\delta^T F^T X^T, \delta = \varepsilon^{-1}. \]

Since

\[ XGAX^T = \begin{bmatrix} G_{11} & G_{12} G_{21}^T & G_{12} G_{21}^T \\ G_{12} G_{22} & G_{22} G_{21} & G_{22} G_{21} \\ G_{12} G_{22} & G_{22} G_{21} & G_{12} G_{21} \end{bmatrix}, \]

and the two submatrices in the last column of \( XGAX^T \) are the same. Then without loss of generality we assume that \( G \) has the following form:

\[ G = \begin{bmatrix} G_{11} & G_2 \\ G_{21} & G_2 \end{bmatrix}. \tag{19} \]

We also assume that \( F \) has the following form:

\[ F = \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} \begin{bmatrix} \lambda_1 G_2 \\ \lambda_2 G_2 \end{bmatrix}, \]

where \( \lambda_1 \) and \( \lambda_2 \) are two scalars to be searched. We suppose \( P \) has the following blocked form:

\[ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \]

corresponding to the blocks of \( G \) and \( F \). Substituting the above expressions of matrices \( P_i G_i F_i \) into (17) and letting \( G_2 A_k = S_A, G_2 B_k = S_B, C_k = S_C, \) and \( D_k = S_D, \) we get (16) from Corollary 3.1 for the robust dichotomy stabilization conditions under uncertainty set (14).

**Remark 3.6:** From the proof of the above theorem, we can see that the assumption of \( G \) with the form of (19) does not introduce any conservativeness. The main conservativeness of the theorem stems from the assumption of \( F \) and the requirement of a single \( G \) and \( F \) for the uncertainty set. In the following, a concrete extended Chua’s circuit example is provided to illustrate the main results and shows that the PDLF method used in the paper will bring less conservativeness than a single Lyapunov function method.

**Remark 3.7:** The above results also provide a new method to avoid chaos by studying the convergence of bounded solutions, which is different from the existing chaos control methods in the literature; see for example [27]–[29]. Based on this method, the nonexistence of chaotic attractors and limit cycles can be studied systematically in theory for both analysis and synthesis.

### IV. Application to an Extended Chua’s Circuit

Chua’s circuit is one of the most important nonlinear circuit which has been used to study a variety of nonlinear dynamic behavior such as bifurcation and chaos [21], [30]–[33]. A typical Chua’s circuit consists of one inductor (\( L \)), two capacitors (\( C_1, C_2 \)), two linear resistors (\( R_0, R \)), and one piecewise-linear resistor (\( f \)). In this section, we give an extended Chua’s circuits with two nonlinear resistors, shown in Fig. 2. The dynamics of the circuit can be described by system (2) with two inputs and two outputs. We will show that how the results obtained in above sections can be used to ensure robust dichotomy in case of system uncertainties and thus avoid chaotic and periodic oscillations in the system.

The extended Chua’s circuit in Fig. 2 can be described by the following differential equations:

\[
\begin{align*}
\dot{v}_1 &= \frac{1}{C_1} \left[ \frac{v_2 - v_3}{R} - f_1(v_1) \right] \\
\dot{v}_2 &= \frac{1}{C_2} \left[ \frac{v_1 - v_2}{R} + i_3 - f_2(v_2) \right] \\
\dot{i}_3 &= \frac{1}{L} \left( -v_2 - R_0 i_3 \right) 
\end{align*}
\tag{20}
\]

where \( v_1, v_2 \) are the voltages across the capacitors \( C_1 \) and \( C_2 \), respectively, \( i_3 \) is the current through the inductor \( L, R \) and \( R_0 \) are resistors, and \( f_1(v_1) \) and \( f_2(v_2) \) are the current through the two nonlinear resistors which are defined as

\[
\begin{align*}
f_1(v_1) &= G_{11} v_1 + \frac{1}{2} (G_{11} - G_{12}) \left[ |v_1 + E| - |v_1 - E| \right] \\
f_2(v_2) &= G_{21} v_2 + \frac{1}{2} (G_{21} - G_{22}) \left[ |v_2 + E| - |v_2 - E| \right].
\end{align*}
\]

Introducing the dimensionless variables

\[
\begin{align*}
x_1 &= \frac{v_1}{E}, & x_2 &= \frac{v_2}{E}, & x_3 &= \frac{i_3 R}{E}, & \tau &= \frac{t}{C_2 R} \\
\alpha &= \frac{C_2}{C_1}, & \beta &= \frac{C_2 R^2}{L}, & \gamma &= \frac{C_2 R_0}{L} \\
m_{11} &= G_{11} R, & m_{12} &= G_{12} R, & m_{21} &= G_{21} R, & m_{22} &= G_{22} R
\end{align*}
\]

then (20) can be transformed into the following dimensionless form:

\[
\begin{align*}
dx_1 &= k\alpha (x_2 - x_1 - f_1(x_1)) \\
dx_2 &= k(x_1 - x_2 + x_3 - f_2(x_2)) \\
dx_3 &= k(-\beta x_2 - \gamma x_3) 
\end{align*}
\tag{21}
\]

with

\[
\begin{align*}
f_1(x_1) &= m_{12} x_1 + \frac{1}{2} (m_{11} - m_{12}) \left[ |x_1 + 1| - |x_1 - 1| \right] \\
f_2(x_2) &= m_{22} x_2 + \frac{1}{2} (m_{21} - m_{22}) \left[ |x_2 + 1| - |x_2 - 1| \right]
\end{align*}
\]
and $k = \pm 1$. Let
\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad z = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \varphi(z) = \begin{pmatrix} f_1(x_1) \\ f_2(x_2) \end{pmatrix}.
\]
System (21) can be written into the form of (2) with
\[
A = \begin{bmatrix} -k\alpha & k\alpha & 0 \\ k & -k & k \\ 0 & -k\beta & -k\gamma \end{bmatrix},
\]
\[
B = \begin{bmatrix} -k\alpha \\ 0 \\ 0 \end{bmatrix},
\]
\[
C = \begin{bmatrix} -k\alpha & k\alpha & 0 \\ k & -k & 0 \end{bmatrix},
\]
\[
D = \begin{bmatrix} -k\alpha & 0 \\ 0 & -k \end{bmatrix}.
\]

It is obvious that
\[
\mu_{11} = \min(m_{11}, m_{12}) \leq f_1'(x_1) \leq \max(m_{11}, m_{12}) = \mu_{12}, \\
\mu_{21} = \min(m_{21}, m_{22}) \leq f_2'(x_2) \leq \max(m_{21}, m_{22}) = \mu_{22}
\]
when $f_1'$ and $f_2'$ exist.

Fig. 3. Simulation results with $|a| = 2.4, |b| = 2.83$.

In the following, we use the method proposed in the above sections to study robust dichotomy of the circuit when there are parameter uncertainties in the linear part of the system and design a dynamic output feedback controller such that the system is dichotomous for all admissible uncertainties. Supposing $\alpha = 1.3018, \beta = 4.4138, \gamma = 4.179, k = 1, m_{11} = 0.3552, m_{12} = 1.0017, m_{21} = 0.2101, m_{22} = 0.021$ and the state matrix is subject to the following uncertainties:
\[
A = \begin{bmatrix} -k\alpha & k\alpha & 0 \\ k & -k & k \\ 0 & -k\beta + a & -k\gamma + b \end{bmatrix}
\]
with $|a| \leq 2.4, |b| \leq \eta$. $\eta$ is an uncertainty bound to be maximized. This is a two-block structured uncertainty which can be described by a four vertex polytope. Solving the inequalities in Theorem 3.2, we can get a set of feasible solutions with the maximum uncertainty bound $\eta = 2.83$. Simulation results of the four vertex systems with initial values (1, 1, -0.5) are shown in Fig. 3 and confirm the robust dichotomy of the system. While when using condition 1) in Lemma 1, since a single Lyapunov matrix $P$ is required, the maximum uncertainty bound obtained is $\eta = 2.24$. This demonstrates the less conservativeness of the proposed method.

Considering the system subject to the following uncertainties:
\[
A = \begin{bmatrix} -k\alpha - c & k\alpha + c & 0 \\ k & -k & k \\ 0 & -k\beta & -k\gamma \end{bmatrix},
\]
\[
B = \begin{bmatrix} -k\alpha - c \\ 0 \\ 0 \end{bmatrix},
\]
\[
C = \begin{bmatrix} -k\alpha - c & k\alpha + c & 0 \\ k & -k & k \end{bmatrix}
\]
with $|\xi| \leq \eta$. When $G_{21} = G_{22} = 0$, the extended Chua’s circuit is the canonical Chua’s circuit with one nonlinear resistor. Taking $\alpha = 9.35150085$, $\beta = 14.790319805$, $\gamma = 0.016073965$, $k = 1$, $m_{11} = -1.138411196$, $m_{12} = -0.722451121$, the system with exact data ($c = 0$) has been studied in [21] and is known to have a chaotic attractor. Taking $m_{21} = -0.36$, $m_{22} = 0.01$, simulation results with initial data (0.8, 0.1, 0.2) show in Fig. 4 that the system is chaotic. In order to design a dynamic output feedback controller such that the closed-loop system is robust dichotomous for all admissible uncertainties, taking $\lambda_1 = \lambda_2 = 1$, $\kappa = \text{diag}(-108, 162)$ and solving the linear matrix inequalities in Theorem 3.2, we get a set of feasible solutions and the dynamic output feedback controller $K(s)$ can be established with

$$D = \begin{bmatrix} -k\alpha - c & 0 \\ 0 & -k \end{bmatrix}$$

such that the resulting closed loop system is dichotomous with the uncertainty bound $\eta = 1.9014$. For the purpose of demonstration, simulation result with initial value (0.8, 0.1, 0.2, 0.03, 0.02, 0.1, 0.4, 0.2) is given in Fig. 5. This illustrative result coincides with Theorem 3.3 and confirms the nonexistence of chaotic oscillations in the system. Compared the effect of the proposed method with that of [11], the solutions of the closed-loop system are convergent by designing a dynamic output feedback controller, while in [11], the closed-loop system with a static feedback controller may drifted toward infinity in some cases.

V. CONCLUSION

New LMI representation of dichotomy for a class of nonlinear systems has been proposed, and the results are extended to uncertain systems with real convex polytopic uncertainties. The conditions obtained enable the determination of parameter-dependent Lyapunov matrices which make the criterion less conservative than single Lyapunov function method. Based on this LMI approach, a dynamic output feedback controller for robust dichotomy is designed, which ensures the nonexistence of chaotic or periodic oscillations in the system.

The results obtained in this paper have been applied to the stabilization of a chaotic state of an extended Chua’s circuit, in the sense that all the bounded solutions of the system are convergent. It is important to emphasize that this method does not
guarantee the stability of the system since there may exist unbounded solutions in the system. However, on the other hand, dichotomy analysis may be part of a method for obtaining stable controllers by at least excluding the possibility of chaotic and periodic behaviors, and it also provides an effective way to detect whether or not a system has bounded oscillations, which is important from both theoretical and practical perspectives.

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REFERENCES


Ying Yang (M’06) received the B.E. and M.E. degrees in electrical engineering from Yanzhan University, China, in 1994 and 1999, respectively, and the Ph.D. degree in control theory from Peking University, Beijing, China, in 2002. From January 2003 to November 2004, she was a Postdoctoral Researcher at the Department of Mechanics and Engineering Science, Peking University. Since 2004, she has worked at the Department of Mechanics and Engineering Science. She is currently an Associate Professor at the Department of Mechanics and Aerospace Engineering, College of Engineering, Peking University. Her research interests include robust and optimal control, nonlinear systems control, and harmonic control of interconnected systems.

Zhisheng Duan received the M.S. degree in mathematics from Inner Mongolia University, Huhhot, China, and the Ph.D. degree in control theory from Peking University, Beijing, China, in 1997 and 2000, respectively. From 2000 to 2002, he worked as a Postdoctoral Researcher at Peking University. Since 2003, he has been an Associate Professor in the Department of Mechanics and Engineering Science, Peking University. His research interests include robust control, stability of interconnected systems, and frequency-domain methods of nonlinear systems.

Dr. Duan received the 2001 Chinese Control Conference Guan-ZhaoZhi Award.

Lin Huang received the B.S. and M.S. degrees in mathematics and mechanics from Peking University, Beijing, China, in 1957 and 1961, respectively. In 1961, he joined the Department of Mechanics and Engineering Science, Peking University, where he is a Professor of Control Theory. His research interests include stability of dynamical systems, robust control, and nonlinear systems. He has authored three books and authored or coauthored more than 200 papers in the fields of stability theory and control.

Prof. Huang is a member of the Chinese Academy of Sciences.