A GENERALIZATION OF SMOOTH CHUA’S EQUATIONS UNDER LAGRANGE STABILITY*

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In this paper, smooth Chua’s equation is generalized to a higher order system from a special viewpoint of interconnected systems. Simple conditions for Lagrange stability are established. And a detailed Lagrange stable region analysis is given for the canonical Chua’s oscillator. In addition, a new nonlinearly coupled Chua’s circuit that appeared in the recent literature is also discussed and a Lagrange stability condition is presented. Several examples are presented to illustrate the results.

Keywords: Lagrange stability; interconnected systems; smooth Chua’s equation; nonlinearly coupled Chua’s circuit.

1. Introduction

Chua’s circuit, as an ideal paradigm of chaos study, has been studied by many researchers in the last 30 years [Chua, 1994, 1998; Madan, 1993; Shil’nikov, 1993]. In the original Chua’s circuit, a piece-wise linear (PWL) function called Chua’s diode plays an important role. Compared with the PWL nonlinearity, smooth nonlinearity is more desirable from a mathematical perspective, and the characteristics of nonlinear devices in real circuits are always smooth. Hence, Chua’s equation with a cubic nonlinearity was proposed [Hartley, 1989]. Smooth Chua’s equations have also been studied extensively and almost all phenomena found in the PWL version also exist in the cubic version [Huang et al., 1996; Khibnik et al., 1993; Liao & Chen, 1998; Tsuneda, 2005]. Besides single Chua’s circuits, coupled Chua’s circuits were also studied by many researchers [Kapitaniak et al., 1994; Cincotti & Stefano, 2004; Duan et al., 2005b; Imai et al., 2002].

In fact, Chua’s circuits either with PWL nonlinearity or with cubic nonlinearity can be viewed as special Lur’e systems. Global stability of Chua’s circuits can be studied by the canonical methods for absolute stability [Curra & Chua, 1997; Narendra & Taylor, 1973; Park, 1997]. Compared with global stability, Lagrange stability (all solutions are bounded) is a weaker property of dynamical systems. Obviously, Lagrange stability and the estimate of the attracting set are very important for the study of chaotic systems. Analysis and control of Lagrange stability for a class of nonlinear systems with infinite equilibria were studied in [Wang et al., 2006; Leonov et al., 1996]. And the boundedness of a special class of Lur’e systems with stiffening nonlinearities were studied in [Arcak et al., 2002].

This paper is devoted to generalizing smooth Chua’s equations under Lagrange stability from a

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viewpoint of interconnected systems. A class of feedback nonlinear systems such as Chua’s circuits can be viewed as interconnected systems composed of a linear subsystem and a nonlinear subsystem, see [Duan et al., 2004, 2005a] for the study of linear interconnected systems. The rest of this paper is organized as follows. In Sec. 2, simple conditions of Lagrange stability are presented for a special interconnected system. Smooth Chua’s equation can be viewed as a special case. In Sec. 3, Lagrange stability condition is also given for a class of special nonlinearly coupled Chua’s circuits studied in [Cincotti & Stefano, 2004]. In Sec. 4, a detailed parameter region analysis for Lagrange stability is given for the canonical Chua’s oscillator over six parameters. Parameter sets of smooth Chua’s equations presented in [Tsuneda, 2005] are classified by Lagrange stability. Examples are given in Sec. 5 and different oscillating phenomena are shown. The last section concludes the paper.

2. Lagrange Stability of a Special Interconnected Nonlinear System

This paper considers the following dynamical system described by the differential equations

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
A_1 x + A_{12} y \\
A_{21} x + A_2 y + R f(y)
\end{bmatrix},
\]

where \(x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n\), \(y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m\), \(f(y) = (y_{k_1}, \ldots, y_{k_m})^T\), \(A_1, A_{12}, A_{21}, A_2\) and \(R\) are real matrices with compatible dimensions, \(k_i\) are odd numbers and \(k_i \geq 3, 1 \leq i \leq m\).

Obviously, system (1) can be viewed as a special interconnected system composed of a linear subsystem \(\dot{x} = A_1 x\) and a nonlinear subsystem \(\dot{y} = A_2 y + R f(y)\). And (1) can also be viewed as a Lur’e system since the nonlinear functions satisfy the general infinite sector constraints \(y_i y_{k_i} \geq 0\), \(i = 1, \ldots, m\).

**Definition 1.** System (1) is said to be Lagrange stable if each of its solutions is bounded.

By the method of [Arcak et al., 2002], we have the following result for Lagrange stability of (1).

**Theorem 1.** If \(A_1\) is Hurwitz stable, \(R\) is diagonal and \(R < 0\), then system (1) is Lagrange stable.

**Proof.** Take \(V(x, y) = x^T P_1 x(t) + y^T P_2 y(t)\) as a Lyapunov function candidate, where \(P_1\) and \(P_2\) are symmetric matrices. The time derivative of \(V(x, y)\) along any trajectory of system (1) is given by

\[
\dot{V}(x, y) = 2x^T P_1 (A_1 x + A_{12} y) + 2y^T P_2 (A_{21} x + A_2 y + R f(y)).
\]

Obviously, for any scalar \(\lambda > 0\),

\[
\begin{align*}
2x^T P_1 A_{12} y &\leq \frac{1}{\lambda} x^T P_1 A_{12} A_{12}^T P_1 x + \lambda y^T y, \\
2y^T P_2 A_{21} x &\leq \frac{1}{\lambda} x^T A_{21}^T A_{21} x + \lambda y^T P_2 P_2 y.
\end{align*}
\]

Hence, we have

\[
\dot{V}(x, y) \leq x^T \left( P_1 A_1 + A_1^T P_1 + \frac{1}{\lambda} P_1 A_{12} A_{12}^T P_1 + \frac{1}{\lambda} A_{21}^T A_{21} \right) x
\]
\[
+ y^T \left( \lambda I + \lambda P_2 \right) y + 2y^T P_2 R f(y).
\]

Since \(A_1\) is Hurwitz stable, there exist \(\lambda > 0\) and \(P_1 > 0\) such that

\[
P_1 A_1 + A_1^T P_1 + \frac{1}{\lambda} P_1 A_{12} A_{12}^T P_1 + \frac{1}{\lambda} A_{21}^T A_{21} < 0.
\]

On the other hand, we can choose \(P_2\) diagonal and positive definite such that \(P_2 R < 0\) due to the assumption on \(R\). Further, by the assumption on \(f(y)\), there is a scalar \(M\) large enough such that

\[
y^T (\lambda I + \lambda P_2 \mathbf{1} + \lambda P_2 \mathbf{2} + A_2^T P_2) y + 2y^T P_2 R f(y) < 0
\]

when there exists \(y_i\) with \(|y_i| \geq M\), \(1 \leq i \leq m\). By (4) and (5), we know

\[
\dot{V}(x, y) < 0
\]

when there exists \(y_i\) with \(|y_i| \geq M\). It is also possible to estimate the attracting set by the method above. By (4), there also exist scalars \(\epsilon > 0\) and \(\lambda > 0\) such that

\[
x^T \left( P_1 A_1 + A_1^T P_1 + \frac{1}{\lambda} P_1 A_{12} A_{12}^T P_1 + \frac{1}{\lambda} A_{21}^T A_{21} \right) x
\]
\[
< -\epsilon x^T x.
\]

Then we have

\[
\dot{V}(x, y) < -\epsilon (x^T x + y^T y) + y^T (\lambda I + \epsilon I + \lambda P_2 \mathbf{2}) y + 2y^T P_2 R f(y).
\]
Similar to (5), there exists a scalar $M^*$ large enough such that
\[
y^T(\lambda I + \epsilon I + \lambda P_2 P_2 + P_2 A_2 + A_2^T P_2) y + 2y^T P_2 R f(y) < 0
\]
when there exists $y_i$ with $|y_i| > M^*$, $1 \leq i \leq m$.

Denoting
\[
d := \max_{y_i \in [-M^*, M^*], 1 \leq i \leq m} (y^T(\lambda I + \epsilon I + \lambda P_2 P_2 + P_2 A_2 + A_2^T P_2) y + 2y^T P_2 R f(y)),
\]
where $\epsilon, \lambda$ and $P_2$ satisfy (6) and (8), then we get
\[
\dot{V}(x, y) < -\epsilon(x^T x + y^T y) + d
\]
which implies that $\dot{V} < 0$ outside the compact set
\[
\Omega := \{(x, y) | \epsilon(x^T x + y^T y) \leq d\}.
\]

Therefore, system (1) is Lagrange stable and every solution $(x, y)$ converges to the smallest level set of $V$ which includes $\Omega$. 

**Remark 1.** Consider the following canonical smooth Chua’s equation [Huang et al., 1996]
\[
\begin{align*}
\dot{x}_1 &= -\alpha x_1 + \alpha x_2 - \alpha x_1^3, \\
\dot{x}_2 &= x_1 - x_2 + x_3, \\
\dot{x}_3 &= -\beta x_2.
\end{align*}
\]

(12)

Obviously, (12) can be viewed as a special case of (1) with
\[
A_1 = \begin{pmatrix} -1 & 1 \\ -\beta & 0 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} \alpha & 0 \end{pmatrix},
\]
\[
A_2 = -\alpha c, \quad R = -\alpha, \quad x = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}, \quad y = x_1.
\]

By Theorem 1, when $\alpha > 0, \beta > 0$, (12) is Lagrange stable. According to [Huang et al., 1996], there is a chaotic attractor in (12) with parameters $\alpha = 10, \beta = 16, c = -0.143$. Of course, every solution of (12) is bounded at this time. By Theorem 1, there are no any requirements on $A_{12}, A_{21}$ and $A_2$. System (1) can be viewed as a general extension of smooth Chua equation (12). According to (12), one can also see from the forthcoming examples that there are many chaotic solutions in (1) under Lagrange stability.

**Remark 2.** In fact, under the assumptions of the theorem, the Lagrange stability mainly owes to the stiffening property of $f(y)$ as studied in [Arcak et al., 2002]. This special property determines the state $y$ converges to an attracting set when there exists $|y_i|$ large enough.

**Remark 3.** According to the proof of Theorem 1, we can also optimize the attracting set of system (1) when it is Lagrange stable by minimizing $d$ in (9).

Besides the quadratic Lyapunov function in Theorem 1, we can obtain the following less conservative theorem by taking a “quadratic plus integral” Lyapunov function.

**Theorem 2.** Suppose $A_1$ is Hurwitz stable. If $R$ is diagonal stable, i.e. there exists a diagonal and positive definite matrix $Q$ such that $QR + R^T Q < 0$, then system (1) is Lagrange stable.

**Proof.** Take $V(x, y) = x^T P_1 x(t) + 2 \sum_{i=1}^{m} \sigma_i(x) + \sum_{i=1}^{m} f_i(x) d \tau$ as a Lyapunov function candidate, where $P_1$ is a symmetric matrix. The time derivative of $V(x, y)$ along any trajectory of system (1) is given by
\[
\dot{V}(x, y) = x^T P_1 (A_1 x + A_{12} y) + 2 f^T(y)Q(A_{21} x + A_2 y + RF(y)),
\]
where $Q = \text{diag}(q_1, \ldots, q_m)$. Obviously, for any scalars $\lambda_1 > 0$ and $\lambda_2 > 0$, we have
\[
2x^T P_1 A_{12} y \leq \frac{1}{\lambda_1} x^T P_1 A_{12} A_{21} x + \lambda_1 y^T y,
\]
\[
2f^T(y)Q(A_{21} x + A_2 y + RF(y)).
\]

Hence, we have
\[
\dot{V}(x, y) \leq x^T \left( P_1 A_1 + A_1^T P_1 + \frac{1}{\lambda_1} P_1 A_{12} A_{21}^T P_1 + \frac{1}{\lambda_2} A_{21} A_{21}^T P_1 \right)
\]
\[
+ \frac{1}{\lambda_1} x^T P_1 A_{21} x + \lambda_1 y^T y + 2 f^T(y)Q(A_2 y + f^T(y)(QR + R^T Q + \lambda_2 QQ) f(y)).
\]

By the assumption on $R$, there are a diagonal matrix $Q > 0$ and a scalar $\lambda_2 > 0$ such that
\[
QR + R^T Q + \lambda_2 QQ < 0.
\]

On the other hand, since $A_1$ is Hurwitz stable, there exist $\lambda_1 > 0$ and $P_1 > 0$ such that
\[
P_1 A_1 + A_1^T P_1 + \frac{1}{\lambda_1} P_1 A_{12} A_{21}^T P_1 + \frac{1}{\lambda_2} A_{21} A_{21}^T P_1 < 0
\]
(16)
where \( \lambda_2 \) satisfies (15). According to (15) and the assumption on \( f(y) \), there is a scalar \( M \) large enough such that
\[
\begin{align*}
\lambda_1 y^T y + 2 f^T(y)QA_2 y + f^T(y)(QR + R^T Q) + \lambda_2 QQf(y) < 0
\end{align*}
\] (17)
when there exists \( y_i \) with \(|y_i| \geq M, 1 \leq i \leq m\). Combining (16), (17) with (14), we know
\[
\dot{V}(x, y) < 0
\]
when there exists \( y_i \) with \(|y_i| \geq M\). Therefore, system (1) is Lagrange stable.

**Remark 4.** One can also give an estimate of the attracting set by Theorem 2. Although Theorem 1 can be viewed as a special case of Theorem 2, it is possible to give a less conservative estimate of the attracting set by Theorem 1 when \( R \) is diagonal and negative definite.

**Remark 5.** We have discussed Lagrange stability of system (1) in the theorems above. In fact system (1) is a Lur’e system, we can also establish conditions of global stability for (1) by the canonical methods for absolute stability.

### 3. A Class of Nonlinearly Coupled Smooth Chua’s Circuits

The following nonlinearly coupled Chua’s circuit was studied in [Cincotti & Stefano, 2004],
\[
\begin{align*}
\dot{x}_1 &= -\alpha c x_1 + \alpha y_1 - \alpha x_1^3 - \alpha x_2^3 \\
&\quad - \alpha K \frac{(x_1 - x_2)^3}{2}, \\
\dot{y}_1 &= x_1 - y_1 + z_1, \\
\dot{z}_1 &= -\beta y_1, \\
\dot{x}_2 &= -\alpha c x_2 + \alpha y_2 - \alpha x_2^3 - \alpha x_1^3 \\
&\quad + \alpha K \frac{(x_1 - x_2)^3}{2}, \\
\dot{y}_2 &= x_2 - y_2 + z_2, \\
\dot{z}_2 &= -\beta y_2.
\end{align*}
\] (18)

Complex behaviors of this sixth-order circuit including synchronization, antisynchronization and hyperchaos were shown in [Cincotti & Stefano, 2004]. By the method in the section above, we can also give a simple result on Lagrange stability of this coupled circuit. First, (18) can be rewritten as
\[
\begin{align*}
\dot{\xi} &= A_1 \xi + A_{12} \eta, \\
\dot{\eta} &= A_{21} \xi + A_2 \eta + R f(\eta) + R_1 g(\eta),
\end{align*}
\] (19)
where
\[
\begin{align*}
A_1 &= \begin{pmatrix}
-1 & 1 & 0 & 0 \\
-\beta & 0 & 0 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & -\beta & 0
\end{pmatrix}, \\
A_{12} &= \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}, \\
A_{21} &= \begin{pmatrix}
\alpha & 0 & 0 & 0 \\
0 & \alpha & 0 & 0
\end{pmatrix}, \\
A_2 &= \begin{pmatrix}
-\alpha & 0 \\
0 & -\alpha
\end{pmatrix}, \\
R &= \begin{pmatrix}
-\alpha - \frac{\alpha K}{2} & -\alpha + \frac{\alpha K}{2} \\
-\alpha + \frac{\alpha K}{2} & -\alpha - \frac{\alpha K}{2}
\end{pmatrix}, \\
R_1 &= \begin{pmatrix}
\frac{3\alpha K}{2} & 0 \\
0 & \frac{3\alpha K}{2}
\end{pmatrix}, \\
f(\eta) &= \begin{pmatrix}
x_1^3 \\
x_2^3
\end{pmatrix}, \\
g(\eta) &= \begin{pmatrix}
x_1^2 x_2 - x_1 x_2^2 \\
-x_1^2 x_2 + x_1 x_2^2
\end{pmatrix}.
\]

Then by Theorem 2, we have

**Theorem 3.** If \( \alpha > 0, \beta > 0 \) and \( K > 0 \), then system (18) is Lagrange stable.

**Proof.** Obviously, \( R < 0 \) and \( A_1 \) is stable. Take \( V(\xi, \eta) = \xi^T(t)P_1(\xi(t) + 2 \sum_{i=1}^{2} \int_{0}^{\tau} f_i(\tau) d\tau \) as a Lyapunov function candidate, where \( P_1 \) is a symmetric matrix. The time derivative of \( V(\xi, \eta) \) along any trajectory of system (1) is given by
\[
\dot{V}(\xi, \eta) = 2 \xi^T P_1 (A_1 \xi + A_{12} \eta) + 2 f^T(\eta) \\
\times (A_{21} \xi + A_2 \eta + R f(\eta) + R_1 g(\eta)).
\] (20)

Obviously, for any scalars \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), we have
\[
\begin{align*}
2 \xi^T P_1 A_{12} \eta &\leq \frac{1}{\lambda_1} \xi^T P_1 A_{12} A_{12}^T P_1 \xi + \lambda_1 \eta^T \eta, \\
2 f^T(\eta)A_{21} \xi &\leq \frac{1}{\lambda_2} \xi^T A_{21}^T A_{21} \xi + \lambda_2 f^T(\eta) f(\eta).
\end{align*}
\]
Hence, we have
\[
\dot{V}(x, y) \leq \xi^T \left( PA_1 + s_1^T P_1 + \frac{1}{\lambda_1} P_1 A_{12} A_{12}^T P_1 
+ \frac{1}{\lambda_2} A_{21}^T A_{21} \right) \xi + \lambda_1 \eta^T \eta + f^T(\eta)(2R
+ \lambda_2 I) f(\eta) + 2f^T(\eta)(R_1 g(\eta) + A_2 \eta).
\]

(21)

Since \( R < 0 \), there exists a scalar \( \lambda_2 > 0 \) such that
\[
2R + \lambda_2 I < 0.
\]

(22)

On the other hand, the stability of \( A_1 \) guarantees that there exist \( \lambda_1 > 0 \) and \( P_1 > 0 \) such that
\[
P_1 A_1 + A_1^T P_1 + \frac{1}{\lambda_1} P_1 A_{12} A_{12}^T P_1 + \frac{1}{\lambda_2} A_{21}^T A_{21} < 0
\]

(23)

where \( \lambda_2 \) satisfies (22). Further, since \( f(\eta) \) is cubic nonlinearity, (22) guarantees
\[
\lambda_1 \eta^T \eta + f^T(\eta)(2R + \lambda_2 I) f(\eta)
+ 2f^T(\eta)(R_1 g(\eta) + A_2 \eta) < 0
\]

when either \( x_1 \) or \( x_2 \) is large enough. In fact, we can analyze (24) from two aspects. (i) If one of \( x_1 \) and \( x_2 \) is small and the other is large, obviously (24) holds. (ii) If both \( x_1 \) and \( x_2 \) are large, \( g_1(\eta) = -g_2(\eta) \) and \( R_1 > 0 \) guarantee that (24) holds. By (23) and (24), \( \dot{V}(\xi, \eta) < 0 \) when either \( x_1 \) or \( x_2 \) is large enough. Therefore, (18) is Lagrange stable. \( \blacksquare \)

As studied in [Cincotti & Stefano, 2004], when \( x_1 = x_2 \), system (18) reduces to a tight coupled system described in (14) of [Cincotti & Stefano, 2004] which is just a special case of interconnected system (1).

Remark 6. Two Chua’s circuits in system (18) are just connected through nonlinear coupling. Of course, we can also study the effects of linear coupling between two circuits. By the discussion in the section above, we can also establish the Lagrange stability condition when linear connection between two circuits in (18) is involved.

4. An Application to Chua’s Oscillator Over Six Parameters

The following Chua’s oscillator was studied by [Tsuneda, 2005]
\[
\begin{align*}
\dot{x}_1 &= -k_0 x_1 + k_0 x_2 - k_0 (a x_1^2 + b x_1), \\
\dot{x}_2 &= k_0 x_1 - k_0 x_2 + k_0 x_3, \\
\dot{x}_3 &= -k_0 x_2 - k_0 (a x_3^2 + b x_3). \\
\end{align*}
\]

(25)

\[
A_1 = \begin{pmatrix} -k & k \\ -k & -k \end{pmatrix}, \quad A_{12} = \begin{pmatrix} k \\ 0 \end{pmatrix},
\]

\[
A_{21} = \begin{pmatrix} k a \\ 0 \end{pmatrix}, \quad A_2 = -k_0 a - k_0 b,
\]

\[
R = -k_0 a a, \quad x = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}, \quad y = x_1.
\]

20 sets of parameter values for (25) were given in Table 1 of [Tsuneda, 2005]. Testing these parameter sets by Theorem 1, we know that (25) is Lagrange stable with parameter sets C-1, C-2, C-4, C-7, C-11, C-12, C-17, C-19 and C-20 (see Table 1 of [Tsuneda, 2005]). For example, for the parameter set C-4 (Table 1 of [Tsuneda, 2005]): \( \alpha = 143.1037, \beta = 207.34198, \gamma = -3.876721, k = -1, a = -0.0156316049, b = -0.897577333, \) all solutions of (25) are bounded, see Fig. 1 for the boundedness of several bounded solutions.

For other parameter sets C-3, C-5, C-6, C-8, C-9, C-10, C-13, C-14, C-15, C-16 and C-18 (see Table 1 of [Tsuneda, 2005]), although chaotic or periodic solutions were shown in [Tsuneda, 2005]), Theorem 1 fails for these parameter sets and computer simulation shows that system (25) is not Lagrange stable, i.e. there exists unbounded solutions. For example, for the parameter set C-9 (Table 1 of [Tsuneda, 2005]): \( \alpha = -4.08685, \beta = -2, \gamma = 0, k = 1, a = 0.0345416029, b = -1.0936936418, \) the solution of this system is bounded when the initial value is small, and the solution would be dramatically unbounded when the initial value becomes larger, see Fig. 2 for two solutions at \( x(0) = (0.032, 0.013, 0.1)^T \) and \( x(0) = (0.5, 0.13, 0.5)^T \).

In what follows, we give a detailed analysis of the parameter region of Lagrange stability and see the conservativeness of Theorem 1 for system (25). From Figs. 3 to 6, the parameter region of Lagrange stability tested by Theorem 1 is shown in pink color. The region of Lagrange stability, which is tested by numerical analysis and computer simulation but cannot be tested by Theorem 1, is shown in blue color. The parameter region in which there exist unbounded solutions is shown in green color. And chaotic solutions are observed around parameter points shown in brown color in the planar figures of Figs. 3–5. It is hard to show the exact positions of brown points in space, so we do not draw brown points in cubic figures.
By Theorem 1, system (25) is Lagrange stable, if
\[
A_1 = \begin{pmatrix} -k & k \\ -k\beta & -k\gamma \end{pmatrix}
\]
is stable and \( R = -\alpha a < 0 \).
\[
(26)
\]
According to the values of \( k \) in [Tsuneda, 2005], we analyze the parameter region of Lagrange stability in two cases.

**Case 1.** \( k = 1 \). At this time, the condition (26) becomes

\[
1 + \gamma > 0, \quad \beta + \gamma > 0 \quad \text{and} \quad \alpha a > 0.
\]

Obviously, this condition is independent of the parameter \( b \). For fixed \( a, b \) and \( \gamma \), e.g. \( a = 0.0639782341, \ b = -1.1619714342 \) and \( \gamma = 0.0160739649 \) (these parameters are chosen from the parameter set C-1 of [Tsuneda, 2005]), the Lagrange stable region for parameters \( \alpha \) and \( \beta \) tested by Theorem 1 (condition (27)) is shown in Fig. 3(a) (pink color). And system (25) is also Lagrange stable at the boundary \( \alpha = 0, \beta > -\gamma \). The computer simulation shows that there are unbounded solutions in the green region. When \( \gamma \) varies with \( \alpha \) and \( \beta \), the \( \alpha - \beta - \gamma \) parameter space of Lagrange stability is shown in Fig. 3(b). The system is also Lagrange stable in the blue plane \( \beta + \gamma > 0, \alpha = 0, \gamma \geq -1 \) which cannot be tested by Theorem 1.
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(a) $\alpha$ and $\beta$ plane

(b) $\alpha-\beta-\gamma$ space

Fig. 3. The Lagrange stable region with $a = 0.0639782341$, $b = -1.1619714342$.

(a) $\alpha$ and $\beta$ plane

(b) $\alpha-\beta-\gamma$ space

Fig. 4. The Lagrange stable region with $a = -0.0163690525$ and $b = -0.8378208148$.

For fixed parameters $a = -0.0163690525$, $b = -0.8378208148$ and $\gamma = 0.0051631393$ (these parameters are chosen from the parameter set C-6 of [Tsuneda, 2005]), besides the pink region (Lagrange stable region tested by Theorem 1) in Fig. 4(a), the system is Lagrange stable at the boundary $\alpha = 0$, $\beta > -\gamma$ and in the region $\alpha < 0$, $-0.0052 \leq \beta \leq -0.0051631393$. When $\gamma$ varies with $\alpha$ and $\beta$, the $\alpha-\beta-\gamma$ parameter space of Lagrange stability is shown in Fig. 4(b). The blue region includes the plane $\alpha = 0$, $\beta + \gamma > 0$, $\gamma \geq -1$ and the space $\alpha < 0$, $-\gamma - 0.0000368607 \leq \beta \leq -\gamma$, $\gamma \geq 0$.

Case 2. $k = -1$. At this time, the condition (26) becomes

$$1 + \gamma < 0, \beta + \gamma > 0 \text{ and } \alpha a < 0.$$  

(28)

For fixed parameters $a = -0.0156316049$, $b = -0.8975773333$ and $\gamma = -1.161224$ (these parameters are chosen from the parameter set C-12 of [Tsuneda, 2005]), at this time, only the boundary $\alpha = 0$, $\beta > -\gamma$ is in blue color in Fig. 5(a). When $\gamma$ varies with $\alpha$ and $\beta$, the $\alpha-\beta-\gamma$ parameter space of Lagrange stability is shown in Fig. 5(b). The plane $\alpha = 0$, $\beta + \gamma > 0$, $\gamma \leq -1$ is in blue color.
From the analysis above, we can see that the blue region is very small, that is, the conservativeness of Theorem 1 is very little for this system. Even sometimes Theorem 1 is almost necessary and sufficient for the test of Lagrange stability, only a part of the boundary is in blue color. In fact, the blue region often appears when Theorem 1 fails and the linear part $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ in (25) is stable.

Although the parameter $b$ does not influence the Lagrange stability condition of Theorem 1, it influences the stability of $A$. Hence, the parameter $b$ influences the blue region slightly. If Theorem 1 fails and $A$ is unstable, unbounded solutions can be observed easily by computer simulation. In the analysis given above, the parameters $a, b$ and $\gamma$ are chosen from the parameter sets of [Tsuneda, 2005].
In order to analyze the conservativeness of Theorem 1 fairly, we take these parameters freely, e.g. $a = 0.0001, b = 10, \gamma = -2$ and $k = -1$, see Fig. 6(a) for the Lagrange stable region. At this time, the blue region is comparatively larger. And the $\alpha-\beta-\gamma$ space is shown in Fig. 6(b) when $\gamma$ varies. If $b, \gamma$ and $k$ are fixed as above, the larger the parameter $a$ is, the smaller is the blue region. And we have not found chaotic solutions in the region for $\alpha$ and $\beta$ in Fig. 6(a).

5. Examples

Example 1. Consider the following system
\begin{align*}
\dot{x}_1 &= -\alpha cx_1 + x_3 - \alpha x_1^3, \\
\dot{x}_2 &= -\alpha cx_2 + \alpha x_3 - \alpha x_2^3, \\
\dot{x}_3 &= x_1 + x_2 - x_3 + x_4, \\
\dot{x}_4 &= -\beta x_3.
\end{align*}
(29)

Of course, this system can be viewed as an interconnected system described by (1) and in fact it is obtained by extending a cubic nonlinearity from the canonical smooth Chua’s circuit. The smooth Chua’s circuit has a chaotic solution for parameters $\alpha = 10, \beta = 16, c = -0.143$. Also for these parameters, we can see the oscillating solution shown in Fig. 7.

By Theorem 1, system (29) is Lagrange stable for the parameters given above, see Fig. 8 for the boundedness of several solutions.

Example 2. Consider the following system
\begin{align*}
\dot{x}_1 &= -\alpha cx_1 + \alpha x_2 - \alpha x_1^3, \\
\dot{x}_2 &= x_1 - x_2 + x_3, \\
\dot{x}_3 &= -\beta x_2 + x_4, \\
\dot{x}_4 &= -2 x_2.
\end{align*}
(30)

![Fig. 7. The solution of (29) with initial value $x(0) = (-0.08, -0.04, 0.02, 0.017)^T$.](image)
Fig. 8. The boundedness of several solutions of (29).

Fig. 9. The solution of (30) with initial value \( x(0) = (-0.08, -0.04, 0.02, 0.017)^T \).
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Fig. 10. The boundedness of several solutions of (30).

Fig. 11. The boundedness of several solutions of (18).

Fig. 12. Two solutions of (18) with $K = -0.13$. 
This system can also be viewed as an interconnected system described by (1) and in fact it is obtained by extending the linear part of the canonical smooth Chua’s circuit. And for the parameters given in Example 1, we can see the oscillating solution shown in Fig. 9.

By Theorem 1, system (30) is Lagrange stable, see Fig. 10 for the boundedness of several solutions.

**Example 3.** Consider system (18) again. Take \( \alpha = 10, \beta = 16, c = -0.143, K = 0.13 \). By Theorem 3, we know that (18) is Lagrange stable for these parameters, see Fig. 11 for the boundedness of several solutions.

However, if we change \( K \) to \(-0.13\), Theorem 3 fails. And computer simulation shows that (18) has unbounded solutions, see Fig. 12 for two solutions at initial values \((0, 0.012, 0, 0, 0.012, 0)^T\) and \((0, 0.012, 0, 0, 0.12, 0)^T\). From Fig. 12, we see that a bounded solution becomes dramatically unbounded when the initial value \(y_2(0)\) increases a little.

6. **Conclusion**

Lagrange stability and the estimate of the attracting sets are important for the study of chaotic systems. In this paper, we have generalized smooth Chua’s equations from a viewpoint of interconnected systems and established simple conditions for the test of Lagrange stability. With the condition given in this paper, the parameter sets given in [Tsuneda, 2005] can be set into two classes by Lagrange stability of the system. And when the system does not satisfy the Lagrange stability condition, computer simulation shows the existence of unbounded solutions. A detailed analysis of Lagrange stable parameter region has been given for Chua’s oscillator studied in [Tsuneda, 2005]. In addition, a simple Lagrange stability condition has also been established for the nonlinearly coupled Chua’s circuits studied in [Cincotti & Stefano, 2004]. The more exact estimate of the attracting set is a topic for the future.

**References**


