Special decentralized control problems in discrete-time interconnected systems composed of two subsystems

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Abstract

In this paper, some special decentralized control problems are addressed for discrete-time interconnected systems. First it is pointed out that some subsystems must be unstable to ensure stability of the overall system in some special cases. Then a special kind of decentralized control problem is studied. This kind of problem can be viewed as harmonic control among independent subsystems. Research results show that two unstable systems can generate a stable system through some effective cooperations. In addition, a decentralized controller design method based on linear matrix inequality is also given by using parameter-dependent Lyapunov function method developed for the study of robust stability. A special linear star coupled dynamical network is also considered. The central subsystem must be unstable to stabilize the whole network under a special coupling. Several examples are given to illustrate the results.

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1. Introduction

When control theory is applied to solving problems of electric power systems, socioeconomic systems, etc., large-scale systems with many state variables often appear. A basic method of dealing with large-scale systems is the decentralized control. Due to its advantages in easier implementation, lower dimensionality and less cost of control laws, decentralized control theory has attracted a great amount of interest since 1970s [10,17]. And in recent years, complex dynamical networks have received a great deal of attention in physical, biological and social sciences, see [11,20,21] and references therein. When every node is considered as a dynamical system, the complex network is a special large-scale system. Of course, the study of decentralized control in complex dynamical networks is also an interesting problem. The main difficulty of solving the decentralized control problem comes from the fact that the feedback gain is subject to structural constraints. Such constraints are of the same nature as the static output ones, which can be viewed as a full state feedback with structural constraints that select only the measured states. After several decades’ of study, the study of necessary conditions for decentralized stabilizability is always based on the concept of fixed mode [3,16,17], which is a natural generalization of the well-known concept of uncontrollable and unobservable modes that appear in the traditional centralized control problems. Many sufficient conditions and methods for decentralized control have also been established based on Lyapunov function method or optimized algorithms, see [8,14,17,18] and references therein. At the beginning the study of large-scale system theory, some people thought that a large-scale system is decentrally stabilizable under controllability condition by strengthening the stability degree of subsystems. Wang [19] showed that this idea is wrong by an example. And because of the existence of decentralized fixed modes, some large-scale systems cannot be decentrally stabilized at all. Generally, it is very conservative that closed-loop subsystems are all required to be stable. Under the stability of subsystems, the actions of interconnections are always ignored and even viewed as disadvantages. This kind of study
is disadvantageous for the study of the actions of interconnections. Along the development of society, interconnections play more and more important roles in social systems, economic systems, power systems, complex networks, etc. However, the study of the effects of interconnections in large-scale systems is still very little to the authors’ knowledge. Recently, some applications of small gain theorem was given in [4] to strengthen robust stability of interconnected systems. In fact, small gain theorem in decentralized control was also used in 1982, see [17, Section 5.1]. And in [17], an example (Example 2.18) of continuous-time system was given to show that some subsystems must be unstable to realize stability of the interconnected system. Duan et al. [6] generalized this kind of problems to a class of special decentralized control problems of continuous-time systems. In such cases, interconnections play real roles for stability of large-scale systems. And the sense of instability was discussed in [1]. The effects of nonlinear input and output coupling was studied in [5]. In this paper, we discuss some special decentralized control problems for discrete-time interconnected systems.

This paper mainly focuses on interconnected systems composed of two subsystems. The results can be generalized to multiple subsystem cases. The rest of this paper is organized as follows. In Section 2, by studying the structure of interconnections we point out that it is impossible to stabilize all subsystems and the whole system simultaneously by using decentralized controllers in some special cases, that is, some subsystems must be unstable to stabilize the whole system. This result shows that the stability of interconnected systems is not only dependent on the stability degree of subsystems, but is closely dependent on the interconnections in some cases. In Section 3, for the sake of studying the effects of interconnections further, we study a special kind of decentralized control problem which can be viewed as harmonic stability problem among independent subsystems. The results show that two unstable subsystems can generate a stable interconnected system. In Section 4, we present an LMI-based decentralized controller design method. In Section 5, a special linear star coupled network is considered. The structural characteristic studied in Section 2 appears during the decentralized controller design for the network. The central subsystem must be unstable to stabilize the whole network under a special coupling. Several examples are given to illustrate the results in Section 6. The last section concludes the paper.

Throughout this paper, det(.) denotes the determinant of the corresponding matrix. Stability of matrix and polynomial means Schur stability. The superscript T means transpose for real matrices.

2. The effects of unstable subsystems

In this paper, we mainly consider the following interconnected system composed of two subsystem:

\[
\begin{align*}
    x_1(k + 1) &= A_{11}x_1(k) + A_{12}x_2(k) + B_1u_1(k), \\
    x_2(k + 1) &= A_{21}x_1(k) + A_{22}x_2(k) + B_2u_2(k),
\end{align*}
\]

where \( u_1(k) = K_1x_1(k) \) and \( u_2(k) = K_2x_2(k) \). We say system (1) is decentrally stabilizable, i.e., there exist \( K_1 \) and \( K_2 \) such that the state matrix of the closed-loop system

\[
A_{cl} = \begin{bmatrix}
    A_1 + B_1K_1 & A_{12} \\
    A_{21} & A_2 + B_2K_2
\end{bmatrix}
\]

is Schur stable.

First, we consider a simple example with the following lemma.

Lemma 1. Given a real monic polynomial \( f(\lambda) \) or a real matrix \( A \), if \( f(\lambda) \) or \( A \) is Schur stable, then \( f(1) > 0 \), or \( \det(I - A) > 0 \).

In system (1), if

\[
\begin{align*}
    A_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & A_{12} &= \begin{pmatrix} z & -z \\ 0 & 0 \end{pmatrix}, \\
    A_{21} &= \begin{pmatrix} \beta & -\beta \\ 0 & 0 \end{pmatrix}, \\
    A_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & B_1 = B_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
    K_1 &= -(k_1 \ k_2), & K_2 &= -(k_3 \ k_4),
\end{align*}
\]

then

\[
A_{cl} = \begin{pmatrix}
    0 & 1 & z & -z \\
    -k_1 & -k_2 & 0 & 0 \\
    \beta & -\beta & 0 & 1 \\
    0 & 0 & -k_3 & -k_4
\end{pmatrix}
\]

Obviously, at this time, \( \det(I - A_{cl}) = (z^2 + k_2z + k_1)(z^2 + k_4z + k_3) - z\beta(z + k_2 + k_1)(z + k_4 + k_3) \). For this simple example, we can get the following results easily:

(i) When \( z\beta = 1 \), 1 is a fixed mode, since \( \det(I - A_{cl}) = 0 \) at this time.

(ii) When \( z\beta > 1 \), for any \( k_1, k_2, k_3, k_4 \) such that \( A_1 + B_1K_1 \) and \( A_2 + B_2K_2 \) are stable, \( A_{cl} \) cannot be stable since \( \det(I - A_{cl}) < 0 \) at this time (Lemma 1).

(iii) When \( z\beta < 1 \), it is possible that the interconnected system and two subsystems can be stabilized simultaneously.

For example, if we take \( z = 1, \beta = 1.5, k_1 = 0.5, k_2 = 0.31, k_3 = -0.76, k_4 = -0.25 \), then we know that \( A_1 + B_1K_1 \) and \( A_{cl} \) are stable, but \( A_2 + B_2K_2 \) is unstable. At this time, \( A_1 + B_1K_1, A_2 + B_2K_2 \) and \( A_{cl} \) cannot be stable simultaneously.

In order to show the complexity of decentralized control problems, an example of continuous-time system was given in [17] (Example 2.18) to show that some subsystems must be unstable to stabilize the interconnected system. A special structural characteristic for such continuous-time systems was established in [6]. According to the example discussed above, we establish a special structural property for such discrete-time systems.
Theorem 1. If the interconnectioned system in (1) satisfies that

\begin{enumerate}
\item there exists \( A'_1 \) such that \( A_{12} = A'_1(I - A_2) \), and \( A'_{12} B_2 = 0 \),
\item there exists \( A'_1 \) such that \( A_{21} = A'_{21}(I - A_1) \), and \( A'_{21} B_1 = 0 \).
\end{enumerate}

then there are not \( K_1 \) and \( K_2 \) such that \( A_1 + B_1 K_1, A_2 + B_2 K_2 \) and \( A_3 \) are Schur stable simultaneously when \( \det(I - A'_{21} A'_{12}) < 0 \).

Proof. Computing the determinant of \( I - A_{cl} \), one can get \( \det(I - A_{cl}) = \det(I - A_1 - B_1 K_1) \det(I - A_2 - 2B_2 K_2 - A_2'(I - A_1)(I - A_1 - B_1 K_1)^{-1} A_{12}'(I - A_2)) \). Noticing conditions (1) and (2), we know \( \det(I - A_{cl}) = \det(I - A_1 - B_1 K_1) \det(I - A_2 - 2B_2 K_2 - A_2'(I - A_1 - B_1 K_1)(I - A_1 - B_1 K_1)^{-1} A_{12}'(I - A_2 - 2B_2 K_2)) \), that is,

\[ \det(I - A_{cl}) = \det(I - A_1 - B_1 K_1) \det(I - A_2 - 2B_2 K_2) \times \det(I - A'_{21} A'_{12}). \]

By Lemma 1, when \( A_1 + B_1 K_1 \) and \( A_2 + 2B_2 K_2 \) are Schur stable and \( \det(I - A'_{21} A'_{12}) < 0 \), we get \( \det(I - A_{cl}) < 0 \), so \( A_{cl} \) is unstable. \( \square \)

Remark 1. Obviously, under the conditions of Theorem 1, when \( \det(I - A'_{21} A'_{12}) = 0 \), \( I \) is a fixed mode. When \( \det(I - A'_{21} A'_{12}) > 0 \), it is possible that there exist \( K_1 \) and \( K_2 \) such that \( A_1 + B_1 K_1, A_2 + 2B_2 K_2 \) and \( A_3 \) are Schur stable simultaneously. And if \( A_{21} \) is stable, there must be one of \( A_1 + B_1 K_1 \) and \( A_2 + 2B_2 K_2 \) is unstable when \( \det(I - A'_{21} A'_{12}) < 0 \). For the study of the effects of interconnections in large-scale systems, it is important to design decentralized controllers when some subsystems must be unstable. At this time, the interconnections play real roles for stability of large-scale systems. In addition, we should also notice that the conditions in Theorem 1 are restrictive. These conditions are related to the computation of the determinant of \( I - A_{cl} \). From this theorem, we can imagine that there exist other cases in which some subsystems must be unstable to stabilize the overall system. How to generalize Theorem 1 is a problem worth to be studied further.

Remark 2. Although the decentralized control law is very economical and easy to implement since the exchange of state information among subsystems is not necessary, it is very hard to design. And Theorem 1 shows the complexity in decentralized control problems. In addition, because of the existence of unstable systems under stability of interconnected systems, the traditional structural disturbance [17], i.e., communication failure between subsystems, is not allowed. Robustness analysis is a basic problem in control theory. For interconnected systems discussed above, we can analyze its robustness under parametric uncertainty in terms of the parameter-dependent Lyapunov method introduced in [12,13].

Corollary 1. For any interconnected matrix \( A = \begin{pmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{pmatrix} \), if \( A_{12} \) and \( A_{21} \) can be written as \( A_{12} = A'_1(I - A_2), A_{21} = A'_2(I - A_1) \), and \( \det(I - A'_{21} A'_{12}) < 0 \), then \( A_1, A_2 \) and \( A \) cannot be Schur stable simultaneously.

Obviously, when \( I - A_1 \) and \( I - A_2 \) are nonsingular, we can take \( A'_{12} = A'_1(I - A_2)^{-1} \) and \( A'_{21} = A'_2(I - A_1)^{-1} \) directly.

The results above can be generalized to cases of multiple subsystems. For example, for an interconnected system composed of three subsystems, its closed-loop system matrix is given by

\[ A_{cl} = \begin{bmatrix} A_1 + B_1 K_1 & A_{12} & A_{13} \\ A_{21} & A_2 + B_2 K_2 & A_{23} \\ A_{31} & A_{32} & A_3 + B_3 K_3 \end{bmatrix}. \]

Let \( \bar{A}_1 = \begin{bmatrix} A_{21} & A_{12} \\ A_{21} & A_2 \end{bmatrix}, \bar{B}_1 = \text{diag}(B_1, B_2), \bar{A}_{13} = \begin{bmatrix} A_{13} \\ A_{31} \end{bmatrix}, \bar{A}_{31} = \begin{bmatrix} A_{31} \end{bmatrix} \). If the following conditions are satisfied:

\begin{enumerate}
\item there exists \( A'_{13} \) such that \( \bar{A}_{13} = A'_{13}(I - A_3), \)
\item there exists \( A'_{31} \) such that \( \bar{A}_{31} = A'_{31}(I - \bar{A}_1), \) and \( \bar{A}_{31} \bar{B}_1 = 0 \),
\end{enumerate}

then there are not \( K_1, K_2 \) and \( K_3 \) such that \( \bar{A}_1 + \bar{B}_1 \) diag \((K_1, K_2), A_3 + B_3 K_3 \) and \( A_{cl} \) are stable simultaneously when \( \det(I - A'_{31} A'_{13}) < 0 \).

Similar to the case of decentralized state feedbacks, we can also study the problem of decentralized dynamical output feedback stabilization. Suppose the outputs of two subsystems in (1) are \( y_1(k) = C_x 1_x(k) \) and \( y_2(k) = C_x 2_x(k) \), respectively. The decentralized dynamical controllers are given as

\[ \begin{align*}
\xi_1(k + 1) &= A F_1 \xi_1(k) + B F_1 y_1(k), \\
\xi_2(k + 1) &= A F_2 \xi_2(k) + B F_2 y_2(k), \\
u_1(k) &= C F_1 \xi_1(k) + D F_1 y_1(k), \\
u_2(k) &= C F_2 \xi_2(k) + D F_2 y_2(k).
\end{align*} \]

Then the state matrix of the closed-loop system is

\[ \tilde{A}_{cl} = \begin{bmatrix} \tilde{A}_1 + \tilde{B}_1 \tilde{F}_1 \tilde{C}_1 & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_2 + \tilde{B}_2 \tilde{F}_2 \tilde{C}_2 \end{bmatrix}, \]

where

\[ \begin{align*}
\tilde{A}_1 &= \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, & \tilde{B}_1 &= \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}, \\
\tilde{C}_1 &= \begin{bmatrix} C_1 & 0 \\ 0 & I \end{bmatrix}, & \tilde{A}_{12} &= \begin{bmatrix} A_{12} & 0 \\ 0 & 0 \end{bmatrix}, \\
\tilde{F}_1 &= \begin{bmatrix} D F_1 & C F_1 \\ B F_1 & A F_1 \end{bmatrix}, \\
\tilde{A}_2 &= \begin{bmatrix} A_2 & 0 \\ 0 & 0 \end{bmatrix}, & \tilde{B}_2 &= \begin{bmatrix} B_2 & 0 \\ 0 & I \end{bmatrix}, \\
\tilde{C}_2 &= \begin{bmatrix} C_2 & 0 \\ 0 & I \end{bmatrix}, & \tilde{A}_{21} &= \begin{bmatrix} A_{21} & 0 \\ 0 & 0 \end{bmatrix}, \\
\tilde{F}_2 &= \begin{bmatrix} D F_2 & C F_2 \\ B F_2 & A F_2 \end{bmatrix}.
\end{align*} \]
static output feedback control. Obviously, if $A_{12}$ and $A_{21}$ satisfy the conditions of Theorem 1, then there are not decentralized controllers (2), i.e., $F_1$ and $F_2$, such that $A_1 + B_1 F_1 C_1$, $A_2 + B_2 F_2 C_2$ and $A_{cl}$ are stable simultaneously.

The above results show that in some cases the stability of interconnected systems is closely dependent on the interconnections. In order to study the actions of the interconnections between subsystems further, we study a special kind of decentralized control problem which can be viewed as harmonic stability problem of subsystems.

3. A class of special decentralized control problems

Consider the following interconnected system:

\begin{align}
 x_1(k+1) &= A_1 x_1(k) + b_{12} u_{12}(k), \\
 x_2(k+1) &= A_2 x_2(k) + b_{21} u_{21}(k),
\end{align}

where $u_{12}(k) = k_1 x_2(k)$ and $u_{21}(k) = k_2 x_1(k)$. $b_{12}$ and $b_{21}$ are given real vectors. $k_1$ and $k_2$ are real row vectors to be determined. There is information interchange between two subsystems, this means that two systems are cooperating. For this special decentralized control problem, we can get a simple result for its stabilizability when at least one of $A_1$ and $A_2$ is with odd order.

**Theorem 2.** Suppose $(A_1, b_{12})$ and $(A_2, b_{21})$ are controllable and $|\tr(A_1) + \tr(A_2)| < \order(A_1) + \order(A_2)$, and at least one of order$(A_1)$ and order$(A_2)$ is odd, where $\tr\cdot$ and order$(\cdot)$ denote the trace and the order of the corresponding matrix, respectively. Then there are real vectors $k_{12}$ and $k_{21}$ such that $A_{cl} = \begin{pmatrix} A_1 & b_{12} k_{12} \\ b_{21} k_{21} & A_2 \end{pmatrix}$ is Schur stable.

**Proof.** Suppose $(A_1, b_{12})$, $(A_2, b_{21})$ are with the standard controllable model. Let the orders of $A_1$ and $A_2$ be $n$ and $m$, respectively. Set $H_1(z) = k_{21}(zI - A_1)^{-1} b_{12}$, $H_2(z) = k_{12}(zI - A_2)^{-1} b_{21}$, $k_{12} = (\beta_0, \beta_1, \ldots, \beta_{m-1})$, $k_{21} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$, $d_1(z) = \det(zI - A_1)$, $d_2(z) = \det(zI - A_2)$, $k_1(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_{n-1} z^{n-1}$, $k_2(z) = \beta_0 + \beta_1 z + \cdots + \beta_{m-1} z^{m-1}$, then $A_{cl}$ is stable if and only if the feedback system shown in Fig. 1 is stable, i.e., the polynomial $d_{cl}(z) = d_1(z) d_2(z) - k_1(z) k_2(z)$ is stable. Obviously, $d_{cl}(z)$ is a monic polynomial and the coefficient of $z^{n+m-1}$ in $d_{cl}(z)$ is $a = -\tr(A_1 + A_2)$ and its absolute value is less than $n + m$. Let $d_1(z) d_2(z) = z^{n+m} + a z^{n+m-1} + d_0(z)$, then the stability of $d_{cl}(z)$ is completely determined by $d(z) = d_0(z) - k_1(z) k_2(z)$. When at least one of $n$, $m$ is odd, we can choose $d(z)$ arbitrarily such that $d_{cl}(z)$ is stable, and decompose $d(z) - d_0(z)$ into the product of real polynomials $k_1(z)$ and $k_2(z)$. This means that we find real vectors $k_{12}$ and $k_{21}$ such that $A_{cl}$ is stable. This completes the proof.

From the proof of Theorem 2, we can see that when we do not require that $k_{12}$ and $k_{21}$ are real vectors we need not care about the orders of $A_1$ and $A_2$. When the orders of $A_1$ and $A_2$, i.e., $n$ and $m$ are even simultaneously, the degrees of $k_1(z)$ and $k_2(z)$ are odd, but the degree of $k_1(z) k_2(z)$ is even. At this time, one needs choose $d(z)$ such that $d_{cl}(z)$ is stable and $d(z) - d_0(z)$ has at least one real root in order to decompose $d(z) - d_0(z)$ into the product of two real polynomials with odd degrees. For this case, we establish the following lemma first.

**Lemma 2.** Given a real monic polynomial with degree $2n$

\[ f(\lambda) = \lambda^{2n} + a_{2n-1} \lambda^{2n-1} + a_{2n-2} \lambda^{2n-2} + \cdots + a_1 \lambda + a_0, \]

with $|a_{2n-1}| < 2n - 2$. If $f(\lambda)$ has one real root in the open unit disc, then there exists a real stable polynomial

\[ g(\lambda) = \lambda^{2n} + a_{2n-1} \lambda^{2n-1} + b_{2n-2} \lambda^{2n-2} + \cdots + b_1 \lambda + b_0 \]

such that $g(\lambda) - f(\lambda)$ has at least two real roots.

**Proof.** Suppose $\alpha$ is a real root of $f(\lambda)$, $\alpha \in (-1, 1)$. The condition $|a_{2n-1}| < 2n - 2$ guarantees that we can choose a stable polynomial $g(\lambda)$ such that $\alpha$ is also a root of $g(\lambda)$ and the sum of all roots of $g(\lambda)$ is $a_{2n-1}$. At this time, we have $g(\lambda) - f(\lambda) = (\lambda - \alpha) h(\lambda)$, where $h(\lambda)$ is a real polynomial with degree $2n - 3$ which is an odd number. Hence, $h(\lambda)$ has at least one real root. Then $g(\lambda) - f(\lambda)$ has at least two real roots.

With Lemma 2, we can give the following result for the existence of real vectors $k_{12}$ and $k_{21}$ by supposing a necessary condition.

**Theorem 3.** Suppose $(A_1, b_{12})$ and $(A_2, b_{21})$ are controllable and $|\tr(A_1) + \tr(A_2)| < \order(A_1) + \order(A_2) - 2$, order$(A_1)$ and order$(A_2)$ are even numbers, where $\tr\cdot$ and order$(\cdot)$ denote the trace and the order of the corresponding matrix, respectively. If $A_1$ or $A_2$ has at least one real eigenvalue inside the open unit disc, then there are real vectors $k_{12}$ and $k_{21}$ such that $A_{cl} = \begin{pmatrix} A_1 & b_{12} k_{12} \\ b_{21} k_{21} & A_2 \end{pmatrix}$ is Schur stable.

**Proof.** Using the notations in the proof of Theorem 2, by Lemma 2 we can find a stable polynomial $g(\lambda)$ (the sum of all its roots is equal to $\tr(A_1) + \tr(A_2)$) satisfying that $d_1(z) d_2(z) - g(\lambda)$ has at least two real roots. Then we can decompose $d_1(z) d_2(z) - g(\lambda) = k_1(z) k_2(z)$ with real polynomials $k_1(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_{n-1} z^{n-1}$, $k_2(z) = \beta_0 + \beta_1 z + \cdots + \beta_{m-1} z^{m-1}$. That is, we find real vectors $k_{12}$ and $k_{21}$ such that $A_{cl}$ is stable.

**Remark 3.** From the proofs of Theorems 2 and 3, we know that the eigenvalues of $A_{cl}$ can be assigned properly with some simple constraints, especially, when at least one of $m$ and $n$ is odd.
In addition, noticing that the sum of eigenvalues of \( A_{cl} \) is equal to \( \text{tr}(A_1) + \text{tr}(A_2) \), we required \( |\text{tr}(A_1) + \text{tr}(A_2)| < \text{order}(A_1) + \text{order}(A_2) - 2 \) in Theorem 3 because of the possibility of the counteraction between the positive and negative real parts of eigenvalues of \( A_{cl} \) (or the counteraction between the positive and negative real parts of roots of \( g(\lambda) \) in Lemma 2).

**Remark 4.** System (3) can be viewed as cooperative behavior between two subsystems. Two subsystems can be unstable themselves, but they can generate a stable system through intercrossed feedback. Subsystem does not use the information itself, but it uses the other subsystem’s information. That is, they can help with each other to realize stability. Of course, there may be self-feedback in subsystems themselves. We can imagine that under cooperations subsystems need not to be controllable or stabilizable themselves.

If there is self-feedback in subsystems, system (3) can be stated as follows:

\[
\begin{align*}
\dot{x}_1(k+1) &= A_1x_1(k) + b_1u_1(k) + b_{12}u_{12}(k), \\
\dot{x}_2(k+1) &= A_2x_2(k) + b_2u_2(k) + b_{21}u_{21}(k),
\end{align*}
\]

where \( u_{12}, u_{21}, b_{12} \) and \( b_{21} \) are given as in system (3), \( b_1 \) and \( b_2 \) are real vectors with compatible dimensions, \( u_1 = k_1x_1(k) \) and \( u_2 = k_2x_2(k) \). By using Theorems 2 and 3, one can get the following result easily.

**Theorem 4.** If \((A_1, [b_1 b_{12}])\) and \((A_2, [b_2, b_{21}])\) are controllable, \( b_1 \) and \( b_2 \) are not zero vectors simultaneously, then there are real vectors \( k_1, k_{12}, k_2, k_{21} \) such that system (4) is stable, i.e., \( A_{cl} = \begin{pmatrix} A_1 + b_1k_1 & b_{12}k_{12} \\ b_{21}k_{21} & A_2 + b_2k_2 \end{pmatrix} \) is Schur stable.

**Proof.** Obviously, no matter what the orders of \( A_1 \) and \( A_2 \) are, we can choose \( k_1 \) and \( k_2 \) such that \( A_1 + b_1k_1 \) and \( A_2 + b_2k_2 \) satisfy the conditions of Theorem 2 or Theorem 3 when \( b_1 \) and \( b_2 \) are not zero vectors simultaneously. Then we can complete the proof easily. \( \square \)

**Remark 5.** One can see clearly in Theorem 4, \((A_1, b_1)\) and \((A_2, b_2)\) need not to be controllable or stabilizable. Theorem 4 shows that two subsystems with effective control can cooperate easily to stabilize the overall system. The actions of interconnections are shown here to some degree. And obviously, the framework in Theorems 2–4 can be generalized to multi-subsystem cases. And according to the proofs of Theorems 2 and 3, we can establish the corresponding algorithms for the design of \( k_{ij} \) based on polynomial method. In what follows, we discuss feedback controller design by the popular LMI method [2,7].

### 4. PDLF method

LMI methods have played leading roles during the last 20 years in linear systems theory. And [18] presented LMI method for decentralized control of nonlinear systems. Although we have analyzed some special decentralized control problems in the sections above, it is still hard to design decentralized controllers. In what follows, we present LMI-based design method for the problems discussed above based on the PDLF method [12,15]. First we introduce the following lemma to begin this section.

**Lemma 3** (de Oliveira et al. [12]). Given a real matrix \( A \in \mathbb{R}^{n \times n} \), \( A \) is Schur stable if, and only if, there exist a matrix \( P = P^T > 0 \) and any matrices \( V \) such that

\[
\begin{pmatrix}
-P & AV \\
V^T A^T & P - V - V^T
\end{pmatrix} < 0.
\]

One can turn (5) into Lyapunov inequality \( P - APA^T > 0 \) easily by using the well known projection lemma in LMI method. By introducing a new variable \( V \), the products of \( PA \) and \( A^TP \) are relaxed to new products \( AV \) and \( V^TA^T \). \( V \) need not be symmetric and positive definite. In this way Lyapunov matrix \( P \) can be parameter-dependent for the study of robust stability and robust performances [12,13,22]. The case of diagonal blocked matrix \( V \) for decentralized control of discrete-time systems was considered in [13]. Here we discuss upper trigonal constraint of \( V \) for system (1) as follows: lower trigonal constraint can be considered similarly. Sometimes, upper trigonal constraint is less conservative than diagonal constraint. Corresponding to system (1), we suppose

\[
V = \begin{pmatrix} V_1 & \lambda V_1 \\
0 & V_2 \end{pmatrix}, \quad X = \begin{pmatrix} X_1 & \lambda X_1 \\
0 & X_2 \end{pmatrix},
\]

where the dimensions of \( V_1 \) and \( V_2 \) are identical with the ones of \( A_1 \) and \( A_2 \), the dimensions of \( X_1 \) and \( X_2 \) are identical with the ones of \( K_1 \) and \( K_2 \), respectively, and \( \lambda \) is a parameter to be chosen which is introduced to reduce the conservativeness further.

**Remark 6.** For simplicity, we assumed that \( V \) and \( X \) acquire the upper trigonal structure as in (6). In fact, it is only fit for the case of order \( (A_1) = \text{order}(A_2) \). If order \( (A_1) \neq \text{order}(A_2) \), for example, order \( (A_1) < \text{order}(A_2) \), we can add zero blocks in \( V \) and \( X \) as follows to meet such cases:

\[
V = \begin{pmatrix} V_1 & V_{12} \\
0 & V_2 \end{pmatrix}, \quad X = \begin{pmatrix} X_1 & X_{12} \\
0 & X_2 \end{pmatrix},
\]

where \( V_{12} = (\lambda V_1), X_{12} = (\lambda X_1) \). When order \( (A_1) > \text{order}(A_2) \), we assume \( V \) and \( X \) acquire the following lower trigonal structure:

\[
V = \begin{pmatrix} V_1 & 0 \\
V_{21} & V_2 \end{pmatrix}, \quad X = \begin{pmatrix} X_1 & 0 \\
X_{21} & X_2 \end{pmatrix},
\]

where \( V_{21} = (\lambda V_2), X_{21} = (\lambda X_2) \), then we can get the similar result as in the following theorem.

Let

\[
A = \begin{pmatrix} A_1 & A_{12} \\
A_{21} & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\
0 & B_2 \end{pmatrix}, \quad K = \begin{pmatrix} K_1 & 0 \\
0 & K_2 \end{pmatrix}
\]

in system (1). By Lemma 3, one can get the following result for stability of system (1).
Theorem 5. If there are \( P = P^T, V \) and \( X \) with the form (6) such that
\[
\begin{pmatrix}
-V - V^T + P & V^T A^T + X^T B^T \\
A V + B X & -P
\end{pmatrix} < 0,
\]
then there exist diagonal blocked matrix \( K \) as in (7) such that
\( A_{cl} = A + BK \) is stable. At this time, decentralized controllers are given as \( K_1 = X_1 V_1^{-1} \) and \( K_2 = X_2 V_2^{-1} \).

Remark 7. From Theorem 5, one can see that \( P \) is not blocked, and \( V_1 \) and \( V_2 \) are generally not symmetric, of course not positive definite. Intuitively, one can imagine that at this time, \( A_1 + B_1 K_1 \) or \( A_2 + B_2 K_2 \) can be unstable under stability of \( A_{cl} \). This point can be seen clearly from the forthcoming examples. We can also establish some similar results for systems (3) and (4). Similar to the method of [22], there is a parameter to be determined in the theorem above. Observing that for given \( \lambda \), the inequality in Theorem 5 is linear, and hence can be solved by LMI toolbox [7]. \( \lambda \) can be searched by the program \textit{fminsearch} as in [22]. This parameter provides a new degree of freedom to design controllers. Therefore, the upper trigonal constraint of \( V \) discussed above is generally less-conservative than the diagonal constraint of \( V \). We can also see this from the forthcoming example. In addition, this LMI method can also be used to discuss robust stability of system (1) under parametric uncertainty as studied in [12,13].

5. A special linear star coupled network

Recently, complex dynamical networks have received a great amount of interest from various researchers, see [11,20,21] and references therein. If every single node in network is viewed as a dynamical system, then the whole network [11] is a special large-scale system. In this section we consider a special linear star coupled network, see Fig. 2. Suppose that every node in Fig. 2 is a linear discrete-time system as follows:
\[
x_i(k+1) = A_1 x_i(k) + B_1 u_i(k),
\]
where \( x_i \) is the state of the node \( i \), \( u_i \) is the control input and \( A_1 \in \mathbb{R}^{n \times n}, B_1 \in \mathbb{R}^{n \times m} \) are given matrices. With the star coupling as in Fig. 2, \( N \) nodes can generate a network as follows:
\[
x(k+1) = Ax(k) + Bu(k),
\]
where
\[
x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix},
\]
\[
A = \begin{pmatrix} A_1 - (N-1)A_{12} & A_{12} & A_{12} & \cdots & A_{12} \\ A_{12} & A_1 - A_{12} & 0 & \cdots & 0 \\ A_{12} & 0 & A_1 - A_{12} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{12} & 0 & 0 & \cdots & A_1 - A_{12} \end{pmatrix},
\]
\[
B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_1 \end{pmatrix},
\]
\[
u = \begin{pmatrix} u_1 \\ 0 \\ \vdots \\ u_N \end{pmatrix},
\]
\( A_{12} \) is the given coupling matrix with compatible dimension. Here, the first node is the central node. The other nodes are connected to the central node, but there are no connections among themselves. In fact, (9) can be viewed as a special symmetrical interconnected system [9]. In what follows, we discuss stability analysis and synthesis problems for (9).

Theorem 6. System (9) is stable if, and only if, \( A_1, A_1 - A_{12} \) and \( A_1 - N A_{12} \) are stable, simultaneously.

Proof. Take
\[
S_N = \begin{pmatrix} I & I & I & \cdots & 0 \\ I & I & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I & I & I & \cdots & I \end{pmatrix},
\]
then
\[
S_N^{-1} = \begin{pmatrix} I & 0 & 0 & \cdots & 0 \\ -I & I & 0 & \cdots & 0 \\ 0 & -I & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -I & 0 & \cdots & I \end{pmatrix}
\]
Taking a similarity transformation for \( A \), we have

\[
S_N^{-1} A S_N = \begin{pmatrix}
A_1 & (N-1)A_{12} & A_{12} & \cdots & A_{12} \\
0 & A_1 - NA_{12} & -A_{12} & \cdots & -A_{12} \\
0 & 0 & A_1 - A_{12} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_1 - A_{12}
\end{pmatrix}
\]

then

\[
T_N^{-1} A_{cl} T_N = \begin{pmatrix}
A_1 - (N-1)A_{12} + B_1 K_1 & (N-1)A_{12} \\
A_{12} & A_1 - A_{12} + B_1 K_2 \\
0 & 0 \\
\vdots & \vdots \\
0 & 0
\end{pmatrix}
\]

Taking a similarity transformation on \( A_{cl} \), we have

\[
T_N^{-1} A_{cl} T_N = \begin{pmatrix}
I & 0 & 0 & \cdots & 0 \\
0 & I & 0 & \cdots & 0 \\
0 & -I & I & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -I & 0 & \cdots & I
\end{pmatrix}
\]

By this transformation, the theorem holds obviously. \( \square \)

Remark 8. Obviously, the interconnected structure studied in Section 1 can appear in \( A_d \) for the controller design problem discussed above. As mentioned in Theorem 1, if there are \( A'_{12} \) and \( A'_{21} \) such that \( (N-1)A_{12} = A'_{12}(I - A_2 + A_{12}), A'_{12} B_1 = 0 \) and \( A_{12} = A'_{21}(I - A_1 + (N-1)A_{12}), A'_{21} B_1 = 0 \), and \( \det(I - A'_{21} A'_{12}) < 0 \), then in order to stabilize \( A_d \) and \( A_1 - A_{12} + B_1 K_1 \), we have to design \( K_1 \) such that \( A_1 - (N-1)A_{12} + B_1 K_1 \) is unstable. This means that the central subsystem must be unstable to stabilize the star coupled network studied above. It is an interesting problem in complex networks.

\[
A_{cl} = \begin{pmatrix}
A_1 - (N-1)A_{12} + B_1 K_1 & A_{12} \\
A_{12} & A_1 - A_{12} + B_1 K_2 \\
0 & 0 \\
\vdots & \vdots \\
A_{12} & 0
\end{pmatrix}
\]

Theorem 7. \( A_{cl} \) is stable if, and only if, \( A_1 - A_{12} + B_1 K_2 \) and

\[
A_d = \begin{pmatrix}
A_1 - (N-1)A_{12} + B_1 K_1 & (N-1)A_{12} \\
A_{12} & A_1 - A_{12} + B_1 K_2 \\
0 & 0 \\
\vdots & \vdots \\
0 & 0
\end{pmatrix}
\]

are stable.

Proof. Similar to the proof of Theorem 6, take

\[
T_N = \begin{pmatrix}
I & 0 & 0 & \cdots & 0 \\
0 & I & 0 & \cdots & 0 \\
0 & I & I & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & I & 0 & \cdots & I
\end{pmatrix}
\]

6. Examples

Example 1. Consider system (1) defined by matrices

\[
A_1 = A_2 = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \quad B_1 = B_2 = \begin{pmatrix}
0 \\
1
\end{pmatrix}.
\]

\[
A_{12} = \begin{pmatrix}
0.8 & -0.8 \\
-0.3 & 0.3
\end{pmatrix}, \quad A_{21} = \begin{pmatrix}
2 & -2 \\
2.5 & -2.5
\end{pmatrix}.
\]

Obviously, the conditions of Theorem 1 are satisfied. Using the method in Theorem 5, after a simple parameter searching as in [22], we can get \( \lambda = -1.5 \) and decentralized controllers as

\[
K_1 = (-0.3658, 1.1579), \quad K_2 = (1.1608, -0.1333)
\]
such that $A_{cl}$ is stable in (1). At this time, $A_2 + B_2K_2$ is not stable. In this example, if we take $\lambda = 0$, the corresponding LMI fails. This demonstrates the less conservativeness of Theorem 5 than the general diagonal blocked constraint of $V$.

**Example 2.** Consider system (3) defined by matrices

\[
A_1 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0.2 & 0.2 & -1
\end{pmatrix}, \quad b_{12} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\
A_2 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0.4 & -0.4 & 1.1
\end{pmatrix}, \quad b_{21} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

Obviously, the conditions in Theorem 2 are satisfied. $A_1$ and $A_2$ are not stable here. Using the method of Theorem 5 and taking $\lambda = 1$, we can get decentralized controllers

\[
k_{21} = (0.3070 \quad 0.2492 \quad -2.1971),
\]

\[
k_{12} = (0.0931 \quad -0.0458 \quad 0.2820)
\]

such that $A_{cl}$ in (3) is stable. In this example, two subsystems are all unstable.

**Example 3.** Consider system (9) again. Let

\[
A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0.8 & -0.8 \\ 0 & 0 \end{pmatrix}.
\]

We design a controller for (9) as discussed in the section above. Taking $N = 10$ for this example, obviously the structural characteristic given in Theorem 1 holds for $A_d$ in Theorem 7 (Remark 8). If we take $K_1 = (-5.78 \ 5.5)$ and $K_2 = (-0.3 \ 0.69)$, then the closed-loop matrix $A_{cl}$ is Schur stable, that is, $A_d$ and $A_1 - A_{12} + B_1K_2$ are Schur stable. However, the central matrix $A_1 - (N - 1)A_{12} + B_1K_1$ is unstable.

7. Conclusion

This paper is mainly devoted to studying some special decentralized control problems in discrete-time interconnected systems. The results in Sections 2 and 3 have shown that some unstable subsystems can generate a stable system under effective interconnections. Even in some special cases, some subsystems must be unstable to stabilize the overall system. This kind of result is obviously important for understanding the actions of interconnections in large-scale systems. Further, an LMI-based decentralized controller design method which is suitable for the problems discussed in this paper has been given by using parameter-dependent Lyapunov function method. Because of the existence of unstable subsystems under stability of the overall system, the traditional structural disturbance in large-scale systems [17] is not allowed for the problems discussed in this paper. At this time, we can discuss robust stability of system (1) by the LMI method introduced in Theorem 5 as studied in [13].

A simple framework has been established for the study of the roles of interconnections in large-scale systems. The results of this paper can be generalized to cases of multiple subsystems. And we have also studied a special star coupled network. Under a specific coupling, the central subsystem must be unstable to stabilize the whole network. Some further generalization to complex networks is of importance in the field of decentralized control and large-scale systems. Although a special structural property has been given in Theorem 1, the conditions are restrictive. And the controller design method given in Theorem 5 is still very conservative. It is obviously an interesting topic to develop more effective decentralized controller design method.

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References


