Synchronization in a class of weighted complex networks with coupling delays

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\textbf{A B S T R A C T}

It is commonly accepted that realistic networks can display not only a complex topological structure, but also a heterogeneous distribution of connection weights. In addition, time delay is inevitable because the information spreading through a complex network is characterized by the finite speeds of signal transmission over a distance. Weighted complex networks with coupling delays have been gaining increasing attention in various fields of science and engineering. Some of the topics of most concern in the field of weighted complex networks are finding how the synchronizability depends on various parameters of the network including the coupling strength, weight distribution and delay. On the basis of the theory of asymptotic stability of linear time-delay systems with complex coefficients, the synchronization stability of weighted complex dynamical networks with coupling delays is investigated, and simple criteria are obtained for both delay-independent and delay-dependent stabilities of the synchronization state. Finally, an example is given as an illustration testing the theoretical results.

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\textbf{1. Introduction}

The complex networks have been gaining increasing recognition as a fundamental tool in understanding dynamical behavior and the response of real systems coming from different fields such as biology, social systems, linguistic networks, and technological systems [1–7]. The dynamics of complex networks has been extensively investigated, with special emphasis on the interplay between the complexity in the overall topology and the local dynamical properties of the coupled nodes. As a typical kind of dynamics, synchronization in complex networks has become of significant interest in recent years. Of particular interest is how the synchronization ability depends on various parameters of the network, such as average distance, clustering coefficient, coupling strength, degree distribution and weight distribution. The dependence of the emergent collective phenomena on the coupling strength and on the topology was unveiled for homogeneous and heterogeneous complex networks [8]. A somewhat surprising finding is that a scale-free network, while having smaller network distances than a small-world network of the same size, is actually more difficult to synchronize [9]. It is shown in Ref. [10] that in the presence of some proper gradient fields, scale-free networks can be more synchronizable than homogeneous networks. The average degree of the network is the key to synchronization and, under certain conditions,
scale-free networks can indeed be synchronized more easily as compared with homogeneous networks when the coupling strength for a given node from other connected nodes (incoming coupling strength) in the network is determined by the local degree of this node [11,12]. For a given network with identical node dynamics, it is shown that two key factors influencing the network synchronizability are the network inner linking matrix and the eigenvalues of the network topological matrix [13]. The synchronizability of weighted aging scale-free networks with non-normalized asymmetrical coupling matrices can be dramatically affected by the asymmetrical parameter, and it can be improved when the couplings from older to younger nodes become dominant [14]. It is shown that the synchronizability of weighted complex networks with a large minimum degree is determined by two leading parameters: the mean degree and the heterogeneity of the distribution of node intensity [15].

Spreading delay of the information through the complex networks is ubiquitous in nature, technology, and society because of finite signal transmission times, switching speeds, and memory effects [16]. Hence, the synchronization of complex networks with delayed coupling has been studied extensively by means of the theoretical and numerical methods. For example, the stability criterion of synchronization in oscillator networks with small-world interactions and coupling delays was derived [17]. Results showed that the stability of synchronization is independent of the network topology. On the basis of the linear matrix inequality or stability theory of the delay systems, some new criteria of synchronization stability in the symmetric networks with coupling delays were obtained for both delay-independent and delay-dependent cases [18–21]. The influence of network topology, connectivity and delay times on synchronization of delayed-coupled chaotic logistic maps was investigated recently. It is shown that when the delay times are sufficiently heterogeneous, the synchronization behavior is largely independent of the network topology but depends on the network connectivity [22].

Importantly, thus, synchronization stability of the weighted complex networks with coupling delays has seldom been analytically investigated. In the present paper, on the basis of the theory of asymptotic stability of linear time-delay systems, the novel criteria of the synchronization stability are derived. The rest of the paper is organized as follows. In Section 2, stability criteria of synchronization for weighted complex dynamical networks with coupling delays are established. A numerical example is given in Section 3, and the conclusion is presented in Section 4.

2. Criteria of synchronization stability for complex networks with coupling delays

We consider a complex dynamical network consisting of \( N \) identically coupled nodes with each node being an \( n \)-dimensional dynamical system, and introduce the coupling delays in this network. The resultant dynamical system can be described as

\[
\dot{x}_i = F(x_i) + c \sum_{j=1}^{N} G_{ij} \Gamma(x_j(t - \tau)), \quad i = 1, 2, \ldots, N
\]

where \( F: \mathbb{R}^n \to \mathbb{R}^n \) is continuously differentiable, \( x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbb{R}^n \) are the state variables of node \( i \). The constant \( c \) is the coupling strength, \( \Gamma = \text{diag}(\tau_1, \tau_2, \ldots, \tau_n) \in \mathbb{R}^{n \times n} \) is a constant 0–1 matrix linking the coupled variables, \( G = (G_{ij})_{N \times N} \) is the coupling configuration matrix of the network and it is not necessarily symmetric, which represents the weighted connection. At the same time, \( G_{ij} \) satisfies

\[
G_{ii} = - \sum_{j=1, j \neq i}^{N} G_{ij} \quad (i = 1, 2, \ldots, N).
\]

The above assumptions can ensure that the completely synchronized state

\[
M = \{x_i = s, \forall i | s = F(s)\},
\]

is an invariant manifold of Eq. (1).

Let \( x = (x_1, x_2, \ldots, x_N), F(x) = (F(x_1), F(x_2), \ldots, F(x_N)) \); we can rewrite system (1) in a compact form as follows:

\[
\dot{x} = \tilde{F}(x) + cG \otimes \Gamma x,
\]

where \( \otimes \) is the direct product.

In what follows, we suppose that the matrix \( G \) is diagonalizable, namely, there exists a nonsingular matrix, \( \Phi = (\phi_1, \phi_2, \ldots, \phi_N) \), such that \( GD\Phi = \mu_k \phi_k \) \((k = 1, 2, \ldots, N)\), where \( \mu_k \) \((k = 1, 2, \ldots, N)\), are the eigenvalues of \( G \). From the above assumptions, it is seen that one of the eigenvalues of \( G \) is zero (set \( \mu_1 = 0 \)).

This paper is mainly aimed at this case, when the synchronous state is a stable equilibrium point. Hence, the synchronization manifold can be set as \( s(t) = e \), where \( e \) is the stable equilibrium state. Clearly, the stability of the synchronized states (3) of the network (1) is determined by the coupling strength \( c \), the inner-coupling matrix \( \Gamma \), the outer-coupling matrix \( G \), and the time-delay constant \( \tau \).

**Lemma 2.1.** Consider the delayed dynamical network (1), whose synchronization manifold is a stable equilibrium state \( s(t) = e \). If the following \( N - 1 \) pieces of \( n \)-dimensional linear delayed differential equations are asymptotically stable about their zero solutions:

\[
\dot{w}(t) = f(e)w + c\mu_i \Gamma w(t - \tau), \quad i = 2, \ldots, N,
\]

where \( f(e) = DF(e) \) and \( DF(e) \) is the Jacobian of \( F(x(t)) \) at \( e \), then the synchronized states (3) are asymptotically stable.
Proof. To investigate the stability of the synchronized states, let
\[ x_i = e + \eta_i(t) \]  
(6)
and, then, we can get the variational equation of Eq. (4),
\[ \dot{\eta} = J(e)\eta + cG \otimes \Gamma \eta \]  
(7)
where \( \eta = (\eta_1, \eta_2, \ldots, \eta_N) \), and \( \eta^T = (\eta_1^T, \eta_2^T, \ldots, \eta_N^T) \) with \( \eta_k^T = \eta_k(t - \tau), k = 1, 2, \ldots, N. \)

By diagonalizing \( G \), this leaves us with a block diagonalized variational equation with each block having the form
\[ \dot{\eta}_k = J(e)\eta_k + c\mu_k \Gamma \eta_k^T \]  
(8)
where \( \mu_k \) is an eigenvalue of \( G \), \( k = 1, 2, \ldots, N. \)

It is clear that we have transformed the stability problem of the synchronized states to the stability problem of the \( n \)-dimensional linear delayed differential equations (8).

Since \( \mu_1 = 0 \) corresponds to the synchronizing state \( e \), the synchronized states are asymptotically stable when the \( N - 1 \) pieces of \( n \)-dimensional linear delayed differential equations are asymptotically stable about their zero solutions:
\[ \dot{w}(t) = J(e)w + c\mu_i \Gamma w, \quad i = 2, \ldots, N. \]  
(9)
The proof is thus completed. \( \square \)

Throughout the whole paper, we mainly study the case when both matrices \( J(e) \) and \( c\mu_i \Gamma \) are commutable, and the matrix \( J(e) \) is diagonalizable. In order to establish the stability criteria of systems (5), we firstly introduce the following preliminaries.

Lemma 2.2. Let \( A \) and \( B \) be diagonalizable \( n \times n \) matrices. \( A \) and \( B \) commute if and only if they are simultaneously diagonalizable, namely, there exists a single inverse matrix \( S \) such that \( S^{-1}AS \) and \( S^{-1}BS \) are simultaneously diagonal.

Consider the following differential systems:
\[ \dot{x} = Ax(t) + Bx(t - \tau) \]  
(10)
where both \( A \) and \( B \) are \( N \times N \) matrices. \( x \in \mathbb{R}^N \)

Theorem 2.3. Let \( A \) and \( B \) commute and each matrix be diagonalizable in systems (10). If the zero solutions of all the following systems:
\[ \dot{y}_j = p_j y_j(t) + q_j y_j(t - \tau), \quad j = 1, 2, \ldots, N, \]  
(11)
are asymptotically stable for \( j = 1, 2, \ldots, N \), then the system (10) is asymptotically stable, where \( p_j \) and \( q_j \) are eigenvalues of the matrices \( A \) and \( B \), respectively.

Proof. Since \( A \) and \( B \) are commutable, and each of them is diagonalizable, there exist a single inverse matrix \( S \) such that \( S^{-1}AS \) and \( S^{-1}BS \) are simultaneously diagonal in terms of Lemma 2.2. Hence, we can let \( D_1 = S^{-1}AS \) and \( D_2 = S^{-1}BS \), respectively, where \( D_1 = \text{diag}(p_1, p_2, \ldots, p_N) \) and \( D_2 = \text{diag}(q_1, q_2, \ldots, q_N) \).

Let \( y(t) = S^{-1}x(t); \) then we have
\[ \dot{y} = D_1 y(t) + D_2 y(t - \tau). \]  
(12)
Thus, systems (12) can be rewritten in component forms as follows:
\[ \dot{y}_j = p_j y_j(t) + q_j y_j(t - \tau), \quad j = 1, 2, \ldots, N. \]  
(13)
Hence, stability of the systems (10) is transformed into the stability problem of the \( N \) pieces of the first-order linear delayed systems (13).

The proof is completed. \( \square \)

Firstly, if \( p_j \) and \( q_j \) are real simultaneously, then the following lemma can be used to determine the stability of the system (13). We consider the following first-order delay differential equation:
\[ \dot{x} = px(t) + qx(t - \tau) \]  
(14)
where \( p, q \in \mathbb{R} \) and \( \tau > 0 \).

Lemma 2.4 ([23]). Suppose \( p \) and \( q \) are real.
1. If \( q^2 < p^2 \), then the zero solution of (14) is asymptotically stable for all \( \tau > 0 \).
2. If \( q^2 > p^2 \) and \( p + q < 0 \), then the zero solution of (14) is asymptotically stable when \( \tau < \tau_0 \) and unstable when \( \tau > \tau_0 \), where \( \tau_0 = \frac{\theta}{2}, \Omega = \sqrt{q^2 - p^2}, \theta = \arccot(\frac{p}{q}). \)
However, if \( p \) (or \( q \)) is complex, we rewrite the system (14) as follows:

\[
\dot{x} = (p_1 + ip_2)x(t) + (q_1 + iq_2)x(t - \tau)
\]

(15)

Let

\[
B = \begin{bmatrix}
(q_1 \cos p_2 \tau + q_2 \sin p_2 \tau) e^{-p_1 \tau} & (q_2 \cos p_2 \tau - q_1 \sin p_2 \tau) e^{-p_1 \tau} \\
-(q_2 \cos p_2 \tau - q_1 \sin p_2 \tau) e^{-p_1 \tau} & (q_1 \cos p_2 \tau + q_2 \sin p_2 \tau) e^{-p_1 \tau}
\end{bmatrix}.
\]

For the stability of the zero solution of the system (15), the results obtained are as follows:

**Lemma 2.5** ([24]). Suppose that \( \text{Re}(p) < -|q| \); then the zero solution of (15) is asymptotically stable for all \( \tau > 0 \).

**Lemma 2.6** ([24]). Let \( p_1 < 0 \). If

\[
2 \sqrt{\det(B)} \sin(\tau \sqrt{\det(B)}) < -\text{tr} B < \frac{\pi}{2\tau} + \frac{2\tau \sqrt{\det(B)}}{\pi}
\]

(16)

and

\[
0 < \tau^2 \det(B) < \left(\frac{\pi}{2}\right)^2
\]

(17)

then the zero solution of (15) is asymptotically stable.

**Lemma 2.6** which includes information on the delay is referred to as the delay-dependent stability criterion for the system (15).

On the basis of the above lemmas, we will formulate the following stability criteria of synchronization for complex networks with coupling delays.

Assume that all eigenvalues of \( J(e) \) are \( \lambda_1, \lambda_2, \ldots, \lambda_n \), and matrix \( J' \) is diag(1, 1, ..., 1, 0, ..., 0) with rank(\( J' \)) = \( r \).

**Theorem 2.7.** If the coupling strength \( c < \left(\frac{\max_{i=1,2,...,n} \text{Re}(\lambda_i)}{\max_{i=1,2,...,n} |\mu_i|}\right) \), then synchronization state (3) is asymptotically stable for any delay.

**Proof.** On the basis of the Theorem 2.3, the stability of synchronization state (3) is transformed into that of the following set of linear delayed systems:

\[
y_{ij} = \lambda_{ij} y_j(t) + c \mu_{ij} y_j(t - \tau), \quad i = 2, \ldots, n, \quad j = 1, 2, \ldots, r
\]

(18)

\[
y_{ij} = \lambda_{ij} y_j(t), \quad j = r + 1, r + 2, \ldots, n.
\]

(19)

Since we have supposed that the equilibrium state \( e \) is stable, the zero solution of systems (19) is asymptotically stable for any delay and \( c \). For systems (18), according to Lemma 2.5, it is inferred that when \( \text{Re}(\lambda_i) < -|\mu_i| \) for \( i = 2, \ldots, n \), and \( j = 1, 2, \ldots, r \), the zero solution of systems (18) is asymptotically stable. By a simple deduction, it is shown that if \( c < \left(\frac{\max_{i=1,2,...,n} \text{Re}(\lambda_i)}{\max_{i=1,2,...,n} |\mu_i|}\right) \), then the zero solution of systems (18) is asymptotically stable for any delay. Hence, if \( c < \left(\frac{\max_{i=1,2,...,n} \text{Re}(\lambda_i)}{\max_{i=1,2,...,n} |\mu_i|}\right) \), synchronization state (3) is asymptotically stable for any delay. \( \square \)

From the systems (18) and (19), it is shown that stability of synchronization state (3) is changed into that of the zero solutions of system (18). In what follows, if \( \lambda_j \) (or \( \mu_i \)) is complex, we let \( \lambda_j = \alpha_j + \beta_j \) and \( \mu_i = \epsilon_i + \delta_i \). Hence, systems (18) can be rewritten as

\[
y_{ij} = (\alpha_j + \beta_j) y_j(t) + c(\epsilon_i + \delta_i) y_j(t - \tau).
\]

(20)

If \( c^2 \mu_i^2 > \lambda_j^2 \) holds for real numbers \( \lambda_j \) and \( \mu_i \) in the system (18), then we let:

(a) \( \tau < \tau_0 \), where \( \tau_0 = \min_{i,j} \frac{\alpha_j^2}{\mu_i^2} \), \( \Omega = \sqrt{c^2 \mu_i^2 - \lambda_j^2} \), \( \Theta = \arccot(\frac{\lambda_j}{\Omega}) \).

And if \( \lambda_j \) (or \( \mu_i \)) is complex, then we have the following conditions;

(b) \( 2 \sqrt{\det(B)} \sin(\tau \sqrt{\det(B)}) < -\text{tr} B < \frac{\pi}{2\tau} + \frac{2\tau \sqrt{\det(B)}}{\pi} \);

(c) \( 0 < \tau^2 \det(B) < \left(\frac{\pi}{2}\right)^2 \), where \( \det(B) = \exp(-2\alpha_j \tau)c^2(\epsilon_i^2 + \delta_i^2) \) and \( \text{tr} B = 2c \exp(-\alpha_j \tau)(\epsilon_i \cos \beta_j \tau + \delta_i \sin \beta_j \tau) \).

**Theorem 2.8.** If the conditions (a), (b) and (c) are satisfied, then synchronization state (3) is asymptotically stable.
Theorem 2.8 can immediately be proved by means of the Lemmas 2.4 and 2.6.
Obviously, the stability criterion obtained includes information on the size of the delay; therefore it can be a delay-dependent criterion of synchronization stability. It is a more general result for testing the synchronization of the weighted complex networks with the coupling delays.

In what follows, some remarks are made as follows. If the matrix $G$ is symmetric and the coupling strength $c$ is complex ($c = c_1 + c_2i$), then we know that the stability of synchronization state (3) is equivalent to that of the following systems:

$$\dot{w}(t) = J(e)w + (c_1 + c_2i)\mu_i, \quad i = 2, \ldots, N,$$

where $\mu_i$ is real.

For this case, the following synchronization criteria can be established by using discussions similar to the above.

Theorem 2.9. If the coupling strength $\sqrt{c_1^2 + c_2^2} < \left(\frac{-\max_{\nu=1,2,\ldots,n} \text{Re}(\lambda_\nu)}{\max_{\nu=1,2,\ldots,n} |\mu_\nu|}\right)$, then synchronization state (3) is asymptotically stable for any delay.

Theorem 2.10. If the following conditions are satisfied:

(a) $2\sqrt{\det(B)} \sin(\sqrt{\det(B)}) B < -\text{tr} B < \frac{-\pi}{\tau} + \frac{2\sqrt{\det(B)}}{\tau}$,

(b) $0 < \tau^2 \det(B) < \left(\frac{\pi}{\tau}\right)^2$, where $\det(B) = \exp(-2\alpha_1\tau)\mu_1^2(c_1^2 + c_2^2)$ and $\text{tr} B = 2\mu_i \exp(-\alpha_i\tau)(c_1 \cos \beta_i \tau + c_2 \sin \beta_i \tau)$ for $i = 2, 3, \ldots, N$ and $j = 1, 2, \ldots, N$,

then synchronization state (3) is asymptotically stable, where $\alpha_j + \beta_ji = \lambda_j$.

3. An illustrative example

Example. Consider the network consisting of the third-order smooth Chua’s circuits [25], in which each node equation is

$$\begin{align*}
\dot{x}_1 &= -k\alpha x_1 + \alpha(x^3_1 + bx_1), \\
\dot{x}_2 &= k\alpha x_1 - k\alpha x_2 + k\alpha x_3, \\
\dot{x}_3 &= -k\beta x_2 - k\gamma x_3.
\end{align*}$$

(22)

Linearizing (22) at its zero equilibrium gives

$$J(e) = \begin{pmatrix}
-k\alpha - k\alpha b & k\alpha & 0 \\
k & -k & k \\
0 & -k\beta & -k\gamma
\end{pmatrix}.
$$

(23)

Now, we take $k = 1$, $\alpha = -0.1$, $\beta = -1$, $\gamma = 1$, $a = 1$, $b = -25$. From simple calculations, it is known that all eigenvalues of $J(e)$ are $\lambda_1 = -0.0208$, $\lambda_2 = -2.1896 + 0.1207i$ and $\lambda_3 = -2.1896 - 0.1207i$. Hence, $J(e)$ is stable, i.e., the single system (22) is locally stable about zero.

The inner-coupling matrix is $G = \text{diag}(1, 1, 1)$. The outer-coupling matrix

$$G = \begin{pmatrix}
-2 & 1 & 0 & 0 & 1 \\
2 & -4 & 1 & 0 & 1 \\
0 & 1 & -2 & 1 & 0 \\
1 & 1 & 2 & -5 & 1 \\
0 & 0 & 1 & 1 & -2
\end{pmatrix}
$$

with all eigenvalues $\mu_1 = 0$, $\mu_2 = -2.2007 + 0.7180i$, $\mu_2 = -2.2007 - 0.7180i$, $\mu_4 = -5.0000$ and $\mu_5 = -5.5987$.

According to Theorem 2.7, if $c < \left(\frac{-\max_{\nu=1,2,\ldots,n} \text{Re}(\lambda_\nu)}{\max_{\nu=1,2,\ldots,n} |\mu_\nu|}\right)$ it is shown that for any delay, the synchronization of the complex network can be achieved. By simple calculations, we can get $c < 0.0208/5.5987 \approx 0.0037$. For clarity, we take the coupling strength $c = 0.0035 < 0.0037$ and time delays $\tau = 2, 30$. The numerical results in Fig. 1 show that networks can eventually achieve the synchronization state at $e = 0$ irrespective of the size of the time delay.

However, if $c^2\mu_j^2 > \lambda_j^2$ holds for real numbers $\lambda_j$ and $\mu_j$ or $\lambda_j$ (or $\mu_j$) is complex, then we can resort to Theorem 2.8 for determining the synchronization stability of the complex networks studied.

Now, we take $c = 1$. By means of some rigorous computation, it is shown that $c^2\mu_j^2 < \lambda_j^2$ holds for all combinations of real $\lambda_j$ and $\mu_j$ in the systems (18). Hence, in these cases, the system (18) is stable. If $\lambda_j$ or $\mu_j$ is complex in the system (18), according to Theorem 2.8, it can be verified that when $\tau = 0.1$, (b) and (c) can be satisfied. Hence, the stability of the synchronization state is achieved for the network consisting of the third-order smooth Chua’s circuits. Fig. 2 shows that synchronization of the complex network is eventually realized as the time is evolving.
Fig. 1. (a) Time series of the first variable $x_{1i}$ of node $i$ for time delay $\tau = 2$. (b) Time series of the first variable $x_{1i}$ of node $i$ for time delay $\tau = 30$. Here, the coupling strength $c = 0.0035$.

Fig. 2. Time series of the first variable $x_{1i}$ of node $i$ for time delay $c = 1$ and $\tau = 0.1$.

4. Conclusion

The synchronization in a class of weighted complex networks with coupling delays was investigated in this paper. On the basis of the stability theory of the linear time-delay system with real and complex coefficients, we have obtained new stability criteria for the synchronization state in the weighted complex dynamical networks with coupling delays. By means of these criteria, we can estimate the range of delay in which synchronization stability can be achieved. Furthermore, it is found that these criteria are easy to verify by means of a combination of theoretical analysis and numerical computation, especially the criteria for the delay-independent stabilities of the synchronization state. Moreover, these synchronization conditions are applicable to networks with different topologies and different sizes. As an illustration, a simple example supports the theoretical analysis.

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