Disconnected Synchronized Regions of Complex Dynamical Networks
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Abstract—This technical note addresses the synchronized region problem, which is converted to a more convenient matrix stability problem, for complex dynamical networks. For any natural number $\tau$, the existence of a network with $\tau$ disconnected synchronized regions is theoretically proved and numerically demonstrated. This shows the intrinsic complexity of the network synchronization problem. Convexity characteristic of stability for relevant matrix pencils is further discussed. A smooth Chua’s circuit network is finally discussed as an example for illustration.

Index Terms—Matrix pencil, network synchronization, synchronized region.

I. INTRODUCTION

The subject of network synchronization has recently attracted increasing attention from various fields (see [2], [3], [6], [9], [19], [21]–[23], [25] and references therein). A similar topic in multi-agent systems and consensus problems has also been studied [13], [14]. In particular, switching topology and time delays were discussed in [14]. Of particular interest is how the synchronizability depends on various structural parameters of the network, such as average distance, clustering coefficient, coupling strength, etc. Some important results have been established for such problems by introducing the notions of master stability function and synchronized region [1], [6], [11], [16], [26]. It is natural to expect achieving strong synchronizability at small cost [12]. It is now well-known that a key factor influencing the network synchronizability is the characterization of the network synchronized region, as discussed in [6], [8], [10], [16]. Obviously, the larger the synchronized region, the easier the synchronization. Some examples for the existence of two or three disconnected synchronized regions were demonstrated in [8]. This technical note attempts to explore the existence of multiple disconnected synchronized regions for various complex networks.

Consider a dynamical network consisting of $N$ coupled identical nodes, with each node being an $n$-dimensional dynamical system, described by

$$\dot{x}_i = f(x_i) + c \sum_{j=1}^{N} a_{ij} H(x_j), \quad i = 1, 2, \ldots, N$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ is the state vector of node $i$, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector-valued function, constant $c > 0$ represents the coupling strength, $H(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is called the inner linking function, and $A = (a_{ij})_{N \times N}$ is the called the outer coupling matrix, which represents the coupling configuration of the entire network. If the entries of $A$ satisfy $a_{ii} = -\sum_{j=1,j\neq i}^{N} a_{ij}, \quad i = 1, 2, \ldots, N$, then network (1) is called a diffusively coupled network. Further if $A$ is symmetric and irreducible, i.e., the graph corresponding to $A$ is undirected and connected, then zero is an eigenvalue of $A$ with multiplicity 1 and all the other eigenvalues of $A$ are strictly negative, which are denoted by $0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N$. The dynamical network (1) is said to achieve (asymptotical) synchronization if $x_1(t) \to x_2(t) \to \cdots \to x_N(t)$, as $t \to \infty$, for which to be useful in engineering applications it is usually expected that all these states converge to, without loss of generality, some synchronous state $s(t) \in \mathbb{R}^n$, an solution of an individual node, i.e., $\dot{s}(t) = f(s(t))$. As shown in [16], the stability of the synchronous solution $s(t)$ can be determined by analyzing the following equation, known as the master stability equation:

$$\dot{\omega} = [DF(s(t)) + \alpha DH(s(t)))] \omega$$

where $\alpha \in \mathbb{R}$ and $DF(s(t))$ and $DH(s(t))$ are the Jacobian matrices of functions $f$ and $H$ at $s(t)$, respectively, where typically $s = s^*$ is an equilibrium (a constant vector). The largest Lyapunov exponent $\lambda_{max}$ of network (1), which can be calculated from system (2) and is a function of $\alpha$, is referred to as the master stability function. In addition, the region $S$ of negative real $\alpha$ where $\lambda_{max}$ is also negative is called the synchronized region of network (1). Based on the results of [16], the stability of the synchronous solution of network (1) is generally determined by the condition

$$\alpha \lambda_k \in S, \quad k = 2, 3, \ldots, N.$$  

The synchronized region $S$ can be an unbounded region, a bounded region, an empty set, or a union of several regions. When the synchronous state is an equilibrium point, then $DF(s(t))$ and $DH(s(t))$ reduce to constant matrices, denoted by $F$ and $H$, respectively. In this case, system (2) becomes $\dot{\omega} = [F + \alpha H] \omega$. Hence, the synchronized region $S$ becomes the stable region of $F + \alpha H$ with respect to parameter $\alpha$. This technical note mainly studies this case when the synchronous state is an equilibrium point.

II. DISCONNECTED STABLE REGIONS FOR MATRIX PENCILS

In this section, the characteristics of disconnected stable regions for the matrix pencil $F + \alpha H$ are studied. In order to discuss this problem in the real parameter domain, the following lemmas are necessary.

Lemma 1: If the real polynomial $p(s) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1 s + \gamma_0, (\gamma_0 > 0)$ is stable, then for any scalar $\epsilon$, $0 < \epsilon < \gamma_0$, the polynomial $p(s) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1 s + \epsilon$ is stable.

Proof: Given that $p(s)$ is stable, polynomial $p_\epsilon(s)$ is stable if and only if $p(s) - \epsilon$ is stable for all $0 < \epsilon < \gamma_0$, or equivalently, the function $\{\epsilon | p(s)/\{1-(\epsilon/p(s))\}\}$ is stable. Further, this is equivalent to that the Nyquist plot of $\{-\epsilon/p(s)\}$ does not enclose the point $(-1, 0)$ for all $0 < \epsilon < \gamma_0$, which obviously holds.

Lemma 2: Given a polynomial $p(\alpha) = (\alpha + 1)(\alpha + 2) \cdots (\alpha + n)$ with variable $\alpha$ and $n \geq 2$, there is a scalar $\beta$ such that $p(\alpha - \beta n)$ has negative (n) real roots.

Proof: Take $\beta > 0$ such that $\beta^2 < (1/2)(0.5 \times 1.5 \times \cdots \times (n/2 - 0.5))$. Then, one can get $p(0) - \beta^2 > 0, p(-1) - \beta^2 > 0, p(-2) - \beta^2 > 0, \cdots, p(-2 + [n/2] - 0.5) - \beta^2 > 0, p(-n) - \beta^2 > 0$. Therefore, the sign of $p(\alpha) = \beta^2$ changes $n$ times on the negative real axis. This means that $p(\alpha - \beta n)$ has $n$ real roots on the negative real axis.

Lemma 3: Given two scalars $\beta_0$ and $\beta$ with $\beta > 0$ and $\beta - \beta_0 > 0$, there are scalars $0 < \alpha_1 < \cdots < \alpha_n$ such that $\alpha_1 \alpha_2 \cdots \alpha_n = \beta$ and all roots of $p(\alpha) = (\alpha + \alpha_1)(\alpha + \alpha_2) \cdots (\alpha + \alpha_n) - (\beta - \beta_0)$ are real.
Proof: By the method of Lemma 2, it suffices to prove this lemma by choosing \( \alpha \), with the above constraints such that the sign of \( p(\alpha) \) changes \( n \) times on the real axis.

With the above lemmas, one can get the following results.

**Theorem 1:** For any natural number \( n \), there are matrices \( F \) and \( H \) of order \( 2(n-1) \) such that \( F + \alpha H \) has \( n \) disconnected stable regions with respect to parameter \( \alpha \).

**Proof:** As shown in Lemma 2, one may take \( \beta > 0 \) such that

\[
p(\alpha) = (\alpha + 1)(\alpha + 2) \cdots (\alpha + 2(n-1)) - \beta^2(n-1) = 0
\]

has \( 2(n-1) \) real roots, denoted by \( \beta_1, \beta_2, \ldots, \beta_{2(n-1)} \). Then, take

\[
H = \begin{pmatrix}
0 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -1 \\
1 & 0 & \cdots & 0
\end{pmatrix}
\]

and

\[
F = -\beta I_{2(n-1)} + F_1, \quad \text{where} \quad I_{2(n-1)} \quad \text{is the identity matrix of order} \quad 2(n-1).
\]

Obviously, the characteristic polynomial of \( F + \alpha H \) is

\[
\det(sI - F - \alpha H) = (s + \beta)^2(n-1) + (\alpha - \beta_1)(\alpha - \beta_2) \cdots (\alpha - \beta_{2(n-1)}).
\]

Using (4), one has

\[
\det(sI - F - \alpha H) = (s + \beta)^2(n-1) - \beta^2(n-1) + (\alpha + 2)(\alpha + 1) + 2(n-1).
\]

The constant term in \( \det(sI - F - \alpha H) \) is \( (\alpha + 1)(\alpha + 2) \cdots (\alpha + 2(n-1)) \), which is larger than zero if the parameter \( \alpha \) is located in the following \( n \) regions:

\[
(0, -1), (-2, -3), \ldots, (-2(n-2), -2n + 3),
\]

\[
(2(n-1), -\infty)
\]

and is smaller than zero if \( \alpha \) is located in the regions:

\[
(-1, -2), (-3, -4), \ldots, (-2n + 3, -2(n-1)),
\]

Obviously, by Lemma 1 \( \det(sI - F - \alpha H) \) has \( n \) disconnected stable regions with respect to parameter \( \alpha \), which are contained in the \( n \) regions shown in (5), respectively.

Combining with the discussions in Section I, for any natural number \( n \), Theorem 1 shows the existence of a network which has \( n \) disconnected synchronized regions. However, for a general network, the node equation is given, i.e., \( F \) is given, which can not be chosen arbitrarily. In this case, one may apply the following result with a chosen inner linking matrix \( H \).

**Theorem 2:** For any given real stable matrix \( F \) of order \( n \), suppose \( \det(sI - F) = s^n + \gamma_{n-1} s^{n-1} + \cdots + \gamma_1 s + \gamma_0 \), and every eigenvalue of \( F \) corresponds to only one Jordan form. If there is a scalar \( \beta_0 \neq 0 \) such that \( p(s) = s^n + \gamma_{n-1} s^{n-1} + \cdots + \gamma_1 s + \gamma_0 - \beta_0 \) is stable and \( p(s) \) has \( n \) pairs of complex conjugate eigenvalues, then there exists a real matrix \( H \) such that \( F + \alpha H \) has \( [n/2] + 1 \) disconnected stable regions with respect to parameter \( \alpha \).

**Proof:** First, suppose that there is a scalar \( \beta_0 \) such that \( p(s) = s^n + \gamma_{n-1} s^{n-1} + \cdots + \gamma_1 s + \gamma_0 - \beta_0 \) is stable, with \( n \) real roots denoted by \( \lambda_1, \ldots, \lambda_n \). Following Lemma 3, take scalars \( 0 < \alpha_1 < \cdots < \alpha_n \) such that \( \alpha_1 \alpha_2 \cdots \alpha_n = \gamma_0 \) and all roots of \( (\alpha + \alpha_1)(\alpha + \alpha_2) \cdots (\alpha + \alpha_n) - (\gamma_0 - \beta_0) \) are real, denoted by \( -\beta_1, \ldots, -\beta_n \).

Consequently

\[
p(\alpha) = (\alpha + \alpha_1)(\alpha + \alpha_2) \cdots (\alpha + \alpha_n) ,
\]

\[
-\beta_0 = (\alpha + \alpha_1)(\alpha + \beta_2) \cdots (\alpha + \alpha_n).
\]

Obviously, \( \beta_1, \ldots, \beta_n = \beta_0 \). Furthermore, take

\[
H_0 = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-1 & 0 & \cdots & 0
\end{pmatrix}
\]

Then, \( \det(sI - F_0) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3) \cdots (s - \lambda_n) \) is similar to \( F \), since each eigenvalue of \( F \) corresponds to only one Jordan form. Moreover, \( \det(sI - F_0 - \alpha H_0) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3) \cdots (s - \lambda_n) - (\gamma_0 - \beta_0) + (\alpha + \alpha_1)(\alpha + \alpha_2) \cdots (\alpha + \alpha_n) \). The constant term in \( \det(sI - F_0 - \alpha H_0) \) is \( (\alpha + \alpha_1)(\alpha + \alpha_2) \cdots (\alpha + \alpha_n) \), which is larger than zero if the parameter \( \alpha \) is located in the following \([n/2] + 1\) regions:

\[
(0, -\alpha_1), (-\alpha_2, -\alpha_3), \ldots
\]

and is smaller than zero if \( \alpha \) is located in the regions:

\[
(-\alpha_1, -\alpha_2), (-\alpha_3, -\alpha_4), \ldots
\]

Thus, by Lemma 1, \( F_0 + \alpha H_0 \) has \([n/2] + 1\) disconnected stable regions, which are located in the regions shown in (7). Since \( F_0 \) is similar to \( F \), there exists a nonsingular matrix \( P \) such that \( P^{-1} F_0 P = F \). Therefore, \( H = P^{-1} H_0 P \) is the matrix to be found. And \( F + \alpha H \) has the same stable regions as \( F_0 + \alpha H_0 \).

Then, assume that there are some complex conjugate pairs in \( \lambda_1, \ldots, \lambda_n \). For simplicity, suppose that there is only one pair of complex conjugate eigenvalues, \( \lambda_1 = \xi_1 + \eta_1 i, \alpha_2 = \xi_1 - \eta_1 i \), and \( \lambda_3, \ldots, \lambda_n \) are all real. Similarly to the above proof, take scalars \( 0 < \alpha_2 < \cdots < \alpha_n \) such that \( \alpha_2 \alpha_3 \cdots \alpha_n = \gamma_0 \) and all roots of \( (\alpha + \alpha_2)(\alpha + \alpha_3) \cdots (\alpha + \alpha_n) - (\gamma_0 - \beta_0) \) are real, denoted by \( -\beta_2, \ldots, -\beta_n \). Consequently, \( p(\alpha) = (\alpha + \alpha_2)(\alpha + \alpha_3) \cdots (\alpha + \alpha_n) - (\gamma_0 - \beta_0) \) is real, denoted by \( -\beta_2, \ldots, -\beta_n \). Obviously, \( \beta_2 \cdots \beta_n = \beta_0 \). Furthermore, take

\[
H_0 = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
-1 & 0 & \cdots & 0
\end{pmatrix}
\]

Obviously, \( \det(sI - F_0) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3) \cdots (s - \lambda_n) \) is similar to \( F \), since each eigenvalue of \( F \) corresponds to only one Jordan form. Moreover, \( \det(sI - F_0 - \alpha H_0) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3) \cdots (s - \lambda_n) - (\gamma_0 - \beta_0) + (\alpha + \alpha_1)(\alpha + \alpha_2) \cdots (\alpha + \alpha_n) \). The constant term in \( \det(sI - F_0 - \alpha H_0) \) is \( (\alpha + \alpha_1)(\alpha + \alpha_2) \cdots (\alpha + \alpha_n) \), which is larger than zero if the parameter \( \alpha \) is located in the following \([n/2] + 1\) regions:

\[
(0, -\alpha_1), (-\alpha_2, -\alpha_3), \ldots
\]

and is smaller than zero if \( \alpha \) is located in the regions:

\[
(-\alpha_1, -\alpha_2), (-\alpha_3, -\alpha_4), \ldots
\]

Repeating the process as above, one can complete the proof easily.

**Remark 1:** For simplicity, in both Theorems 1 and 2, the parameter \( \alpha \) only appears in the constant term of the characteristic polynomial of \( F + \alpha H \). In this way, the constant term in \( \det(sI - F - \alpha H) \), which is a polynomial of parameter \( \alpha \), simply determines the disconnected stable regions. If \( \alpha \) appears in the higher-order terms of \( \det(sI - F - \alpha H) \), the problem becomes harder to solve, leaving an interesting topic for further research.
Remark 2: In order to guarantee $H$ to be a real matrix, two cases are considered in the proof of Theorem 2, i.e., there are or there are no complex conjugate pairs in $\lambda_i, i = 1, \cdots, n$. If $H$ can be chosen to be a complex matrix, the proof of Theorem 2 will be simplified and $H$ may be chosen such that $F + \alpha H$ has $[n/2] + 1$ disconnected stable regions. In addition, if all $\lambda_i, i = 1, \cdots, n$, are complex scalars, then there exists a real $H$ such that $F + \alpha H$ has at least $[n/4] + 1$ disconnected stable regions.

According to the above discussions, one can also choose a suitable $H$ such that $F + \alpha H$ has only one convex stable region with respect to parameter $\alpha$, as further discussed below.

III. CHARACTERISTICS OF CONVEXITY FOR STABILITY OF MATRIX PENCILS

In the previous section, it shows the existence of any $n$ disconnected stable regions of the matrix pencil $F + \alpha H$. Contrarily to this non-convexity, given two parameter values $\alpha_1$ and $\alpha_2$, whether or not the stability of $F + \alpha_1 H$ and $F + \alpha_2 H$ implies the stability of $F + (\alpha_1 + (1 - \lambda)\alpha_2)H$, for all $0 \leq \lambda \leq 1$, is an interesting problem. Obviously, a good understanding of this convexity characteristic is useful for enhancing the stability of the matrix pencil $F + \alpha H$.

Lemma 4: Suppose that $F + \alpha_1 H$ and $F + \alpha_2 H$ are stable, and the rank of $H$ is 1. Let $H = bc$, where $b$ is a column vector and $c$ is a row vector with compatible dimensions, and $(F, b)$ be controllable. Then, the following conditions are equivalent to each other:

i) $\lambda(F + \alpha_1 H)^{-1} + (1 - \lambda)(F + \alpha_2 H)$ is stable for all $0 \leq \lambda \leq 1$;

ii) There is a common matrix $P = P^T$ such that $P(F + \alpha_1 H) + (F + \alpha_2 H)^TP < 0, i = 1, 2$;

iii) $(F + \alpha_1 H)(F + \alpha_2 H)$ does not have negative real eigenvalues;

iv) $1 - \text{Re} \{((\alpha_2 - \alpha_1)\rho_{ij} I - F - \alpha_1 H)^{-1}b\} > 0, \forall w \in \mathbb{R}$.

Further, if any one of i)-iv) holds, one has $\{\lambda(F + \alpha_1 H) + (1 - \lambda)(F + \alpha_2 H)\}$ is stable for all $0 \leq \lambda \leq 1$.

Proof: See [18] for the equivalence among ii)-iv). Now, if i) holds, then $\det(\lambda(F + \alpha_1 H)^{-1} + (1 - \lambda)(F + \alpha_2 H)) \neq 0, \forall 0 \leq \lambda \leq 1$. Equivalently, $\det(\{(\lambda - 1)I + (F + \alpha_1 H)(F + \alpha_2 H)\}) \neq 0, \forall 0 \leq \lambda \leq 1$, which implies iii). On the other hand, by a simple congruence transformation with matrix $(F + \alpha_1 H)^{-1}$, ii) implies the existence of a common matrix $P = P^T$ such that

$$P(F + \alpha_1 H)^{-1} + (F + \alpha_1 H)^{-T}P < 0$$

which implies i). And, obviously, ii) implies (x). This completes the proof.

Remark 3: Any one of i)-iv) in Lemma 4 implies (x). However, generally, (x) does not imply the other conditions of Lemma 4. For example, with $F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha_1 = 0, \alpha_2 = 0.9$, obviously Lemma 4 (x) holds for the above matrices, but all i)-iv) in Lemma 4 do not hold. In addition, obviously Lemma 4 ii) is equivalent to (8). However, since the rank of $(F + \alpha H)^{-1} - (F + \alpha_2 H)$ is generally not 1, (8) is not equivalent to that $(F + \alpha H)^{-1} - (F + \alpha_2 H)$ does not have negative real eigenvalues.

Remark 4: It is not necessary to require $P > 0$ in Lemma 4 ii). The positive definiteness of $P$ is naturally guaranteed by the stability of $F + \alpha H, i = 1, 2$. Actually, for given matrices $F$ and $P = P^T$ of order $n$, if $PFP + PTQ < 0$ then $F$ has no eigenvalues on the imaginary axis, $\det(P) \neq 0$, and the number of positive eigenvalues of $P$ is equal to the number of eigenvalues of $F$ with negative real parts [7].

Remark 5: To the instability of matrix pencils, a similar result can be obtained from Lemma 4 and Remark 4. Suppose that $F + \alpha_1 H$ and $F + \alpha_2 H$ are unstable and do not have imaginary eigenvalues, and the rank of $H$ is 1. Let $H = bc$ and $(F, b)$ be controllable. Then, the conditions of Lemma 4 (ii)-(iv) are equivalent to each other. Further, if any one of Lemma 4 ii)-iv) holds, then $F + (1 - \lambda)\alpha_1 H + \lambda\alpha_2 H$ is unstable for all $\lambda \in (0, 1)$.

The common Lyapunov matrix problem for companion stable matrix pencils was studied in [17], [18]. Remark 5 above generalizes similar results to unstable matrix pencils. In fact, any one of ii)-vi) in Lemma 4 guarantees that transferring an eigenvalue between the left-half and right-half complex planes is impossible. Therefore, when any one of ii)-vi) in Lemma 4 holds, $F + \alpha_1 H$ and $F + \alpha_2 H$ must have the same number of eigenvalues with positive real parts.

Although the characteristics of convexity for stability or instability of the matrix pencil $F + \alpha H$ have been discussed when the rank of $H$ is 1, it is still hard to decide the stability or instability of the convex combinations of $F + \alpha_1 H$ and $F + \alpha_2 H$ for a general $H$. The results of Lemma 4 and Remark 5 are directly related to the existence of a common matrix $P$ for two vertex matrices. For a general $H$, this common-matrix method is very conservative. In this case, nevertheless, the following lemma provides a less conservative criterion [15].

Lemma 5: Suppose that $F + \alpha_1 H$ and $F + \alpha_2 H$ are stable. If there are matrices $P_1 = P_1^T, P_2 = P_2^T, G$ and $V$ such that

$$\begin{pmatrix} -G - G^T & P_1 - V^T + G F_{\alpha} \\
G^T P_2 - V + F_{\alpha} G & V F_{\alpha} + F_{\alpha} V^T \end{pmatrix} < 0, \quad i = 1, 2$$

where $F_{\alpha} = F + \alpha, H$, then $F + \lambda\alpha_1 H + (1 - \lambda)\alpha_2 H$ is stable for all $0 \leq \lambda \leq 1$.

Similarly to the method in [4], [15], [24], by introducing new slack matrices $G$ and $V$, the symmetrical matrices $P_1$ and $P_2$ can be chosen parameter-dependent for the study of stability of the convex combination of $F + \alpha_1 H$ and $F + \alpha_2 H$.

Remark 6: For instability of matrix pencils, one can likewise obtain a similar result. Suppose that $F + \alpha_1 H$ and $F + \alpha_2 H$ are unstable. If there are matrices $P_1 = P_1^T, P_2 = P_2^T, G$ and $V$ such that the matrix inequalities in Lemma 5 hold, then $F + (\lambda\alpha_1 + (1 - \lambda)\alpha_2)H$ is unstable for all $0 \leq \lambda \leq 1$.

Remark 7: Section II shows the existence of disconnected stable regions of matrix pencils. However, it is generally hard to determine the real stable regions for given matrices. Section III presents a basic convex method. Just like other stability analysis method such as the Lyapunov theory, this method is generally conservative, i.e., the tested stable regions are usually smaller than the actual stable regions. For example, take $F = \begin{pmatrix} -2 & -1 \\ 4 & -1 \end{pmatrix}$, $H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\det(sI - (F + \alpha H)) \equiv s^2 + 4s + 7\alpha + 2(\alpha + 1)(\alpha + 4)$. By the polynomial method, one knows that $F + \alpha H$ is stable for $\alpha \in (-7, -3)$ and $\alpha \in (-2, 2)$, which are two disconnected regions. By the common Lyapunov method described in Lemma 4, one can get the stable regions $[\lambda = 0, -3], [\lambda = -1, 1.88]$ and $[\lambda = 0, -3]$, which are strictly smaller than the real stable regions. On the other hand, by the parameter-dependent Lyapunov method described in Lemma 5, one can get the stable regions $[\lambda = 0, -3], [\lambda = -1, 1.88]$, which approximate the real stable regions very well. This also shows that this method is less conservative than the common Lyapunov function method. Of course, there also exist other examples for which the method in Lemma 5 cannot find a satisfactory stable region. Actually, the most important feature of the convex method is its efficient controller design. When the network non-communicated regions are disconnected, one can design a common local controller to enlarge the stable region, as studied in [5], [8]. For example, one can add local control to network (1), then the controlled network becomes

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} a_i \dot{H}(x_j) + Bu_i$$
For all $i = 1, 2, \ldots, N$, where $B$ is a given matrix and $u_i = Kx_i$, with $K$ being a node-state-feedback gain matrix to be designed. By the master function method as discussed in Section I, the synchronized region for this controlled network is reduced to the stable region of the matrix $F + \alpha H + BK$ with parameter $\alpha$. With the above matrices $F$ and $H$, one can take $B = (0, 0, 1)^T$. Then based on the method described in Lemma 4, one can get a controller $K = (-2.0499, 0.7661, -8.8091)$ such that $F + \alpha H + BK$ is stable for $\alpha \in [-7, 2]$. This obviously enlarges the size of the stable region.

Remark 8: Other than the convex method discussed above for determining the stable region of $F + \alpha H$, one can also use the structured singular value method [27] to search for this stable region. Let $H = BC$ be a full-rank decomposition of $H$. First, determine a parameter value $\alpha_0$ such that $F + \alpha_0 H$ is stable. Then, determine a maximum value of $\gamma$ such that $F + \alpha_0 H + \epsilon H$ is stable for $|\epsilon| \leq \gamma$. Then, $F + \alpha_0 H + \epsilon H$ is stable if and only if $F + \alpha_0 H + \epsilon H$ has no imaginary eigenvalues for all $|\epsilon| \leq \gamma$, that is

$$det(jwI - F - \alpha_0 H - \epsilon H) \neq 0.$$ 

Because of the stability of $F + \alpha_0 H$, this is equivalent to

$$det(I - \epsilon(jwI - F - \alpha_0 H)^{-1} H) \neq 0,$$

i.e.,

$$det(I - \epsilon C(jwI - F - \alpha_0 H)^{-1} B) \neq 0$$

which is a frequency-related real structured singular value problem [27]. The stability radius $\gamma$ around $\alpha_0$ can be searched along the imaginary axis by the structured singular value method. Finally, change the initial parameter $\alpha_0$ to find the maximum $\gamma$ in a convex stable region. Because the stable region of $F + \alpha H$ can be disconnected, it is clear that this stability problem is intrinsically hard to solve.

IV. SYNCHRONIZATION OF A SMOOTH CHUA’S CIRCUIT NETWORK

Consider the network (1) consisting of the third-order smooth Chua’s circuits [20], in which each node equation is

$$\begin{align*}
\dot{x}_{i1} &= -k_1 x_{i1} + k_2 x_{i2} - k_3 (ax_{i1}^3 + bx_{i1}) \\
\dot{x}_{i2} &= k_1 x_{i1} - k_2 x_{i2} + k_3 x_{i3} \\
\dot{x}_{i3} &= -k_3 x_{i2} - k_7 x_{i3}.
\end{align*}$$

(9)

The vector $x_i$ in (1) is $(x_{i1}, x_{i2}, x_{i3})^T$ here. Linearizing (9) at its zero equilibrium gives $\dot{x} = Fx$, $F = \begin{pmatrix} -k_3 & -k_7 & 0 \\ k & -k & k \\ 0 & -k_3 & -k_7 \end{pmatrix}$.

Take $k = 1, \alpha = -0.1, \beta = -1, \gamma = 1, a = 1, b = -25$. Then $F$ is stable, i.e., the node system (9) is locally stable about zero. One can easily take a parameter $\beta_0 = -0.8$ such that all roots of $det(sI - F - \beta_0)$ are real. Following the method of Theorem 2, take $\alpha_1 = 0.01$, $\alpha_2 = 1$, $\alpha_3 = 10$, and $H = \begin{pmatrix} 0.8348 & 9.6619 & 2.6591 \\ 0.1002 & 0.0694 & 0.1005 \\ -0.3254 & -8.5837 & -0.9042 \end{pmatrix}$.

Then, by simply computation, one knows that $F + \alpha H$ has two disconnected stable regions: $S_1 = [-0.0009, 0]$ and $S_2 = [-2.2255, -1]$. Therefore, the entire synchronized region is $S_1 \cup S_2$. Further, suppose that the number of nodes is $N = 10$, and the outer coupled matrix $A$ is a globally coupled matrix, i.e., all the diagonal entries of $A$ are $-9$ and the other entries are $1$, which has eigenvalues $\lambda_1 = 0$, $\lambda_2 = \cdots = \lambda_{10} = -10$. Then, by (3), network (1) with the above parameter values achieves local synchronization when the coupling strength $\epsilon$ satisfies $\epsilon \in [0, 0.00099]$ or $\epsilon \in [0.1, 0.2225]$. Figs. 1 and 2 show the synchronization and non-synchronization behaviors of this network.

Similarly to the above Chua’s circuit network synchronization problem, for the case of higher-order Chua’s circuits being the network nodes, one can find more disconnected synchronized regions.

V. CONCLUSION

In this technical note, the problem of disconnected synchronized regions has been carefully studied. When the synchronous state is an equilibrium point, the problem is converted to the more convenient stability problem of matrix pencils. The existence of multiple disconnected synchronized regions is theoretically proved for networks with higher-dimensional nodes. This shows a possibility of intermittent synchronization in complex networks when the coupling strength is varied. Further, the characteristic of convexity for matrix pencils has been discussed. Some conditions for testing the stability and instability of convex combinations of vertex matrices have also been established. Finally, a network of smooth Chua’s circuits has been simulated for illustrating the analytic results. It is noted that for a given network, how to determine beforehand whether it has bounded, unbounded
or disconnected synchronized regions remains a technical challenge for future research.

REFERENCES